

Reductive locally homogeneous pseudo-Riemannian manifolds and Ambrose-Singer connections

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Abstract

Ambrose and Singer characterized connected, simply-connected and complete homogeneous Riemannian manifolds as Riemannian manifolds admitting a metric connection such that its curvature and torsion are parallel. The aim of this paper is to extend Ambrose-Singer Theorem to the general framework of locally homogeneous pseudo-Riemannian manifolds. In addition we study under which conditions a locally homogeneous pseudo-Riemannian manifold can be recovered from the curvature and their covariant derivatives at some point up to finite order. The same problem is tackled in the presence of a geometric structure.

1 Introduction

In [1] Ambrose and Singer characterized connected, simply-connected and complete homogeneous Riemannian manifolds as Riemannian manifolds (M, g) admitting a linear connection $\tilde{\nabla}$ satisfying

$$\tilde{\nabla}g = 0, \quad \tilde{\nabla}R = 0, \quad \tilde{\nabla}S = 0,$$

where $S = \nabla - \tilde{\nabla}$, ∇ is the Levi-Civita connection of g , and R the curvature tensor field of g . Connections satisfying the previous equations would become later known as Ambrose-Singer connections. Since their introduction, Ambrose-Singer connections have become an extensively used tool for the study of homogeneity. In addition, Ambrose-Singer Theorem has been extended to the case when the manifold is endowed with a geometric structure [5], and later to the pseudo-Riemannian setting [4].

Regarding locally homogeneous spaces, the following result is known (see for instance [13]).

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Theorem 1.1 *Let (M, g) be a Riemannian manifold. Then (M, g) is locally homogeneous if and only if it admits an Ambrose-Singer connection.*

This theorem is no longer true if g is a metric with signature. In fact, in [4] it was proved that if a globally homogeneous pseudo-Riemannian manifold admits an Ambrose-Singer connection, then it must be reductive. This suggests that in order to extend Ambrose-Singer Theorem to the pseudo-Riemannian setting we have to find a condition playing the same role as the reductivity condition for globally homogeneous spaces.

The aim of this paper is to formulate and prove an analogous result to Theorem 1.1 for pseudo-Riemannian manifolds. The case when an invariant geometric structure is present is also analyzed.

On the other hand, all the proofs of Theorem 1.1 (known by the author) make use of the so called “canonical” Ambrose-Singer connection constructed by Kowalski [8]. The construction of this connection relies strongly on the fact that the Killing form of $\mathfrak{so}(T_p M)$ is definite if the metric g is Riemannian, so that a straightforward adaptation to the pseudo-Riemannian realm is not possible. In Section 5 we show, under suitable conditions, how to adapt the construction of the “canonical” Ambrose-Singer connection made by Kowalski to metrics with signature. This will lead to a new notion of reductivity called “strong reductivity”. As a consequence we will see that strongly reductive locally homogeneous pseudo-Riemannian manifolds can be recovered from the curvature and their covariant derivatives at some point up to finite order. Recall that this property is known to be satisfied by all locally homogeneous Riemannian manifolds (see [10]). An analogous result will hold in the presence of an invariant geometric structure.

2 Preliminaries

For a comprehensive introduction on Lie pseudo-groups and transitive Lie algebras see [12] and the references therein. We just recall that a transitive Lie algebra is a pair (L, L^0) , where L is a Lie algebra and L^0 is a proper subalgebra such that the only ideal of L contained in L^0 is $\{0\}$.

Let (M, g) be a locally homogeneous pseudo-Riemannian space, and let \mathcal{I} denote the Lie pseudo-group of local isometries acting transitively on (M, g) . The system of PDE's that must be satisfied by the elements of \mathcal{I} is

$$f^*g = g.$$

The corresponding system of Lie equations is thus

$$\mathcal{L}_X g = 0, \tag{1}$$

that is, infinitesimal transformations are given by local Killing vector fields. For a fixed point $p \in M$ we choose a basis $\{e_1, \dots, e_m\}$ of $T_p M$. The set $\{e^1, \dots, e^m\}$ denotes its dual basis. We consider the transitive Lie algebra $(\mathfrak{i}, \mathfrak{i}^0)$ associated with the system (1). The Lie algebra \mathfrak{i} is the set of vector valued formal power series

$$\xi = \sum_{r, i, j_1, \dots, j_r} \xi_{j_1 \dots j_r}^i e_i \otimes e^{j_1} \odot \dots \odot e^{j_r},$$

where $\xi_{j_1 \dots j_r}^i$ solve (1) and all its derivatives. The subalgebra \mathfrak{i}^0 is formed by all the elements of \mathfrak{i} such that the terms ξ^i of order zero vanish. As seen in [12], an element $\xi \in \mathfrak{i}$ is determined by the terms of order 0 and 1, which lie in $T_p M$ and $\mathfrak{so}(T_p M)$ respectively.

Definition 2.1 A Killing generator at p is a pair $(X, A) \in T_p M \times \mathfrak{so}(T_p M)$ verifying

$$A \cdot \nabla^i R_p + i_X \nabla^{i+1} R_p = 0, \quad i \geq 0,$$

where ∇ is the Levi-Civita connection of g , and R its curvature tensor field.

The set \mathfrak{kil} of Killing generators at p has a Lie algebra structure with bracket

$$[(X, A), (Y, B)] = (AX - BY, (R_p)_{XY} + [A, B]).$$

We define

$$\mathfrak{kil}^0 = \{(X, A) \in \mathfrak{kil} / X = 0\}.$$

Lemma 2.2 [12] $(\mathfrak{kil}, \mathfrak{kil}^0)$ is a transitive Lie algebra isomorphic to $(\mathfrak{i}, \mathfrak{i}^0)$.

Proof. Let (x^1, \dots, x^m) be a set of normal coordinates around p . We consider the map

$$\begin{aligned} \mathfrak{i} &\rightarrow \mathfrak{kil} \\ (\xi^i, \xi_j^i) &\mapsto (\xi^i \partial_{x^i}|_p, \xi_j^i \partial_{x^i}|_p \otimes dx_p^j), \end{aligned}$$

where (ξ^i, ξ_j^i) are the terms of order 0 and 1 characterizing an element $\xi \in \mathfrak{i}$. As a straightforward computation shows, this map defines a Lie algebra homomorphism. ■

Let now ξ be a local vector field on M , we define the $(1, 1)$ -tensor field

$$A_\xi = \mathcal{L}_\xi - \nabla_\xi = -\nabla \xi,$$

where \mathcal{L} denotes Lie derivative. Among the equations that ξ must satisfy at p , we have

$$(\mathcal{L}_\xi g)_p = 0, \quad (\mathcal{L}_\xi \nabla^i R)_p = 0, \quad i \geq 0,$$

which coincide with

$$A \cdot g_p = 0, \quad A \cdot \nabla^i R_p + i_X \nabla^{i+1} R_p = 0, \quad i \geq 0,$$

for $X = \xi_p$ and $A = A_\xi|_p$, whence $(\xi_p, A_\xi|_p)$ is a Killing generator.

Corollary 2.3 Every formal solution $\xi \in \mathfrak{i}$ is realized by the germ of a local Killing vector field.

Proof. Adapting the arguments used by Nomizu in [11] to metrics with signature, we see that if the dimension of the Lie algebra of Killing generators is constant on M , then for every Killing generator (X, A) at a point p , there exist a local Killing vector field ξ with $(X, A) = (\xi_p, A_\xi|_p)$. ■

The Lie algebra isomorphism exhibited in the proof of Lemma 2.2 can be seen as

$$\begin{aligned} \mathfrak{i} &\rightarrow \mathfrak{kil} \\ [\xi] &\mapsto (\xi_p, A_\xi|_p), \end{aligned}$$

where $[\xi]$ denotes the germ of the local vector field ξ at p .

3 Reductive locally homogeneous pseudo-Riemannian manifolds

We now consider a Lie pseudo-group $\mathcal{G} \subset \mathcal{I}$ acting transitively on (M, g) . A Lie subalgebra $\mathfrak{g} \subset \mathfrak{i}$ can be attached to \mathcal{G} , namely \mathfrak{g} is the set of germs of local Killing vector fields with 1-parameter group contained in \mathcal{G} . The Lie algebra \mathfrak{k} formed by those $[\xi] \in \mathfrak{g}$ vanishing at p is thus a Lie subalgebra of \mathfrak{i}^0 , and the pair $(\mathfrak{g}, \mathfrak{k})$ is a transitive Lie algebra.

Definition 3.1 *Let \mathcal{G} be a Lie pseudo-group of isometries acting transitively on (M, g) . The isotropy pseudo-group at a point p is*

$$\mathcal{H}_p = \{f \in \mathcal{G} / f(p) = p\}.$$

Since $f(p) = p$ is not a differential equation, \mathcal{H}_p is not a Lie pseudo-group in general. For this reason it is more convenient to work with the so called linear isotropy group.

Definition 3.2 *The linear isotropy group of \mathcal{G} at $p \in M$ is*

$$H_p = \{F : T_p M \rightarrow T_p M / F = f_*, f \in \mathcal{H}_p\}.$$

Since every $f \in \mathcal{H}_p$ is an isometry, H_p is a Lie subgroup of $O(T_p M)$.

Lemma 3.3 *The Lie algebra \mathfrak{h}_p of H_p is isomorphic to \mathfrak{k} .*

Proof. We define the map

$$\begin{aligned} \mathfrak{k} &\rightarrow \mathfrak{h}_p \\ [\xi] &\mapsto \left. \frac{d}{dt} \right|_{t=0} (f_t)_*, \end{aligned}$$

where $f_t \in \mathcal{H}_p$ is the 1-parameter group generated by ξ . A simple inspection shows that this map is a Lie algebra isomorphism. ■

Note that seeing \mathfrak{h}_p as a subalgebra of $\mathfrak{so}(T_p M)$, the previous isomorphism between \mathfrak{k} and \mathfrak{h}_p can be read as

$$\begin{aligned} \mathfrak{k} &\rightarrow \mathfrak{h}_p \\ [\xi] &\mapsto A_\xi|_p. \end{aligned}$$

There is a natural action of H_p on \mathfrak{g} given by

$$\begin{aligned} \text{Ad} : H_p \times \mathfrak{g} &\rightarrow \mathfrak{g} \\ (F, [\xi]) &\mapsto [\eta], \end{aligned}$$

with

$$\eta_q = \left. \frac{d}{dt} \right|_{t=0} f \circ \varphi_t \circ f^{-1}(q),$$

for every q in a certain neighborhood of p , where φ_t is the 1-parameter group generated by $[\xi]$, and $F = f_*$. When identifying \mathfrak{k} with \mathfrak{h}_p , the restriction of this action to \mathfrak{k} is just the usual adjoint action of H_p on its Lie algebra.

Definition 3.4 Let (M, g) be a pseudo-Riemannian manifold, and let \mathcal{G} be a Lie pseudo-group of isometries acting transitively on (M, g) . We say that the triple (M, g, \mathcal{G}) is reductive if the transitive Lie algebra $(\mathfrak{g}, \mathfrak{k})$ associated with \mathcal{G} can be decomposed as $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$, where \mathfrak{m} is $\text{Ad}(H_p)$ -invariant.

Note that being reductive is a property of the triple (M, g, \mathcal{G}) rather than a property of the pseudo-Riemannian manifold (M, g) itself. In Section 7 we will show that the same locally homogeneous pseudo-Riemannian manifold can be reductive for the action of a certain Lie pseudo-group \mathcal{G} , whereas it is non-reductive for the action of another Lie pseudo-group \mathcal{G}' . On the other hand, it seems that the previous definition depends on the chosen point $p \in M$, however

Proposition 3.5 If (M, g, \mathcal{G}) is reductive at a point $p \in M$, then it is reductive at every point.

Proof. Let q be another point of M . We denote by $(\mathfrak{g}_p, \mathfrak{k}_p)$ and $(\mathfrak{g}_q, \mathfrak{k}_q)$ the transitive Lie algebras associated with \mathcal{G} at p and q respectively. Let $h \in \mathcal{G}$ be a local isometry with $h(p) = q$. h induces isomorphisms $\tilde{h} : \mathfrak{g}_p \rightarrow \mathfrak{g}_q$, $[\xi] \mapsto [h_*(\xi)]$, and $\tilde{h} : H_p \rightarrow H_q$, $F \mapsto h_* \circ F \circ h_*^{-1}$. Let $\mathfrak{g}_p = \mathfrak{m}_p \oplus \mathfrak{k}_p$ with \mathfrak{m}_p $\text{Ad}(H_p)$ -invariant, we define $\mathfrak{m}_q = \tilde{h}(\mathfrak{m}_p) \subset \mathfrak{g}_q$. It is obvious that $\mathfrak{g}_q = \mathfrak{m}_q \oplus \mathfrak{k}_q$, since \tilde{h} is an isomorphism and takes \mathfrak{k}_p to \mathfrak{k}_q . We now show that \mathfrak{m}_q is $\text{Ad}(H_q)$ -invariant and independent of the local isometry h . Let $F \in H_q$, and let $f \in \mathcal{H}_q$ with $F = f_*$. Let $[\eta] \in \mathfrak{m}_q$, there is an element $[\xi] \in \mathfrak{m}_p$ with $\eta = h_*(\xi)$. The 1-parameter group generated by η is thus $\phi_t = h \circ \varphi_t \circ h^{-1}$, where φ_t is the 1-parameter group generated by ξ . Therefore

$$\begin{aligned} \text{Ad}_F([\eta]) &= \left[\frac{d}{dt} \Big|_{t=0} f \circ \phi_t \circ f^{-1} \right] \\ &= \left[\frac{d}{dt} \Big|_{t=0} f \circ h \circ \varphi_t \circ h^{-1} \circ f^{-1} \right] \\ &= \left[\frac{d}{dt} \Big|_{t=0} h \circ h^{-1} \circ f \circ h \circ \varphi_t \circ h^{-1} \circ f^{-1} \circ h \circ h^{-1} \right] \\ &= h_* \left(\text{Ad}_{\tilde{h}^{-1}(F)}([\xi]) \right). \end{aligned}$$

Since $\tilde{h}^{-1}(F) \in H_p$, we have $\text{Ad}_F([\eta]) \in \mathfrak{m}_q$. On the other hand, in order to prove the independence of h , it is enough to prove that for other $h' \in \mathcal{G}$ with $h'(p) = q$ we have that $h_*^{-1} \circ h'_*([\xi]) \in \mathfrak{m}_p$. But

$$h_*^{-1} \circ h'_*([\xi]) = \left[\frac{d}{dt} \Big|_{t=0} h^{-1} \circ h' \circ \varphi_t \right] = \text{Ad}_{(h^{-1} \circ h')_*}([\xi]).$$

Since $h^{-1} \circ h' \in \mathcal{H}_p$ and \mathfrak{m}_p is $\text{Ad}(H_p)$ -invariant we conclude that $h_*^{-1} \circ h'_*([\xi]) \in \mathfrak{m}_p$. ■

Following [1] we give the following definition.

Definition 3.6 An Ambrose-Singer connection (AS-connection for sort), is a linear connection $\tilde{\nabla}$ satisfying

$$\tilde{\nabla}g = 0, \quad \tilde{\nabla}R = 0, \quad \tilde{\nabla}S = 0,$$

where $S = \tilde{\nabla} - \nabla$.

The following two theorems characterize locally homogeneous pseudo-Riemannian manifolds admitting an AS-connection.

Theorem 3.7 *Let (M, g, \mathcal{G}) be a reductive locally homogeneous pseudo-Riemannian manifold. Then (M, g) admits an AS-connection.*

Proof. Let (r, s) be the signature of g , and let $\mathcal{O}(M)$ be the bundle of orthonormal references of M . We fix a point $p \in M$ and a reference $u_0 \in \mathcal{O}(M)$ in the fiber of p . We shall interpret an orthonormal reference u at $q \in M$ as an isometry $u : (\mathbb{R}^m, \langle \cdot, \cdot \rangle) \rightarrow (T_q M, g_q)$, where $\langle \cdot, \cdot \rangle$ is the standard metric of \mathbb{R}^m with signature (r, s) . Consider the set

$$Q = \{u \in \mathcal{O}(M) / u = \tilde{h}(u_0), h \in \mathcal{G}\}, \quad (2)$$

where \tilde{h} is the map induced on $\mathcal{O}(M)$ by a local isometry h . Q determines a reduction of $\mathcal{O}(M)$ with structure group

$$\bar{H} = \{B \in O(r, s) / \hat{u}_0(B) = f_*, f \in \mathcal{H}_p\},$$

where $\hat{u}_0 : O(r, s) \rightarrow O(T_p M)$, $B \mapsto u_0 \circ B \circ u_0^{-1}$. It is obvious that \hat{u}_0 gives an isomorphism between \bar{H} and the linear isotropy group H_p . The right action of an element $B \in \bar{H}$ on a reference $u \in Q$ at q is given by $R_B(u) = u \circ B : \mathbb{R}^m \rightarrow T_q M$. Let $F = \hat{u}_0(B) \in H_p$ and let $f \in \mathcal{H}_p$ such that $F = f_*$. Let $h \in \mathcal{G}$ be such that $u = \tilde{h}(u_0)$, we can write

$$\begin{aligned} R_B(u) &= u \circ B = u \circ u_0 \circ F \circ u_0 \\ &= h_* \circ u_0 \circ u_0^{-1} \circ F \circ u_0 = h_* \circ f_* \circ u_0 \\ &= \tilde{h} \circ \tilde{f}(u_0). \end{aligned}$$

We now consider the map

$$\begin{aligned} \Psi : \quad \mathfrak{g} &\rightarrow T_{u_0} Q \\ [\xi] &\rightarrow \left. \frac{d}{dt} \right|_{t=0} \widetilde{\varphi}_t(u_0), \end{aligned}$$

where φ_t is the 1-parameter group of ξ . Ψ is injective as $\{\varphi_t\} \subset \mathcal{G}$ and the action of \mathcal{G} on Q is free. Moreover,

$$\dim(\mathfrak{g}) = \dim(T_p M) + \dim(\mathfrak{k}) = \dim(T_p M) + \dim(V_{u_0} Q) = \dim(T_{u_0} Q),$$

whence Ψ is a linear isomorphism. Let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ be a reductive decomposition, we define the horizontal subspace at u_0 as

$$H_{u_0} Q = \Psi(\mathfrak{m}),$$

and making use of \mathcal{G} we define an horizontal distribution on Q as

$$H_u Q = \tilde{h}_*(H_{u_0}), \quad u = \tilde{h}(u_0).$$

This horizontal distribution is \mathcal{C}^∞ and invariant by \mathcal{G} . In order to see that HQ defines a linear connection $\tilde{\nabla}$ on M we just have to show that it is equivariant by the right action of \bar{H} . Let $B \in \bar{H}$, we take $F = \hat{u}_0(B)$, and $f \in \mathcal{H}_p$ with $F = f_*$. Let $X_u \in H_u Q$, by definition $X_u = \tilde{h}_*(X_{u_0})$

for some $X_{u_0} \in H_{u_0}Q$ and some h such that $u = \tilde{h}(u_0)$. This means that $X_u = \tilde{h}_*(\Psi([\xi]))$ for some $[\xi] \in \mathfrak{m}$. Let φ_t be the 1-parameter group generated by ξ , we thus have

$$\begin{aligned}
(R_B)_*(X_u) &= (R_B)_* \circ \tilde{h}_* \circ \Psi([\xi]) \\
&= \left. \frac{d}{dt} \right|_{t=0} R_B \circ \tilde{h} \circ \widetilde{\varphi}_t(u_0) = \left. \frac{d}{dt} \right|_{t=0} \tilde{h} \circ \widetilde{\varphi}_t \circ \tilde{f}(u_0) \\
&= \left. \frac{d}{dt} \right|_{t=0} \tilde{h} \circ \tilde{f} \circ \tilde{f}^{-1} \circ \widetilde{\varphi}_t \circ \tilde{f}(u_0) \\
&= (\tilde{h} \circ \tilde{f})_* \left(\left. \frac{d}{dt} \right|_{t=0} \tilde{f}^{-1} \circ \widetilde{\varphi}_t \circ \tilde{f}(u_0) \right) \\
&= (\tilde{h} \circ \tilde{f})_* (\Psi(\text{Ad}_{F^{-1}}([\xi]))) .
\end{aligned}$$

Since $\text{Ad}_{F^{-1}}([\xi]) \in \mathfrak{m}$, we have $\Psi(\text{Ad}_{F^{-1}}([\xi])) \in H_{u_0}Q$, and we conclude that $(R_B)_*(X_u) \in H_{R_B(u)}Q$ since $R_B(u) = h \circ \tilde{f}(u_0)$.

We now study the properties of $\tilde{\nabla}$. On the one hand, since Q is a reduction of $\mathcal{O}(M)$, the connection $\tilde{\nabla}$ is metric, that is, $\tilde{\nabla}g = 0$. On the other hand, the connection $\tilde{\nabla}$ is characterized in the following way. Let $p, q \in M$, and let γ be a path in M with $\gamma(0) = p$ and $\gamma(1) = q$. We denote by $\tilde{\gamma}$ the horizontal lift of γ to $u_0 \in Q$ with respect to $\tilde{\nabla}$. The parallel transport along γ with respect to this connection is thus the linear isometry $\gamma : T_p M \rightarrow T_q M$ given by $\gamma = u \circ u_0^{-1}$, where $u = \tilde{\gamma}(1)$. But since $u = \tilde{h}(u_0) = h_* \circ u_0$ for some $h \in \mathcal{G}$, we have that the linear isometry γ is exactly h_* . This characterization of $\tilde{\nabla}$ implies that its torsion \tilde{T} and curvature \tilde{R} are invariant by parallel transport, since $\tilde{\nabla}$ is invariant by \mathcal{G} , that is $\tilde{\nabla}\tilde{T} = 0$ and $\tilde{\nabla}\tilde{R} = 0$. As a straightforward computation shows this two equations are equivalent to

$$\tilde{\nabla}R = 0, \quad \tilde{\nabla}S = 0.$$

■

Theorem 3.8 *Let (M, g) be a pseudo-Riemannian manifold admitting an AS-connection $\tilde{\nabla}$. Then there is a Lie pseudo-group of isometries \mathcal{G} such that (M, g, \mathcal{G}) is reductive locally homogeneous.*

Proof. Let $p, q \in M$, we consider a path γ from p to q . Since $\tilde{\nabla}$ is an AS-connection, the parallel transport $\gamma : T_p M \rightarrow T_q M$ with respect to $\tilde{\nabla}$ is a linear isometry preserving the torsion and curvature of $\tilde{\nabla}$. This implies that there exist neighborhoods \mathcal{U}^p and \mathcal{U}^q , and an affine transformation $f^\gamma : \mathcal{U}^p \rightarrow \mathcal{U}^q$ with respect to $\tilde{\nabla}$, such that its differential at p coincides with the parallel transport along γ (see [6, Vol. I, Ch. VI]). Since $\tilde{\nabla}$ is metric we have that f^γ is an isometry. We consider the set

$$\mathcal{G} = \{f^\gamma / \gamma \text{ is a path from } p \text{ to } q\}.$$

\mathcal{G} is a pseudo-group of local isometries of (M, g) which acts transitively on (M, g) , so that (M, g) is locally homogeneous. In addition, \mathcal{G} coincides with the so called transvection group of $\tilde{\nabla}$, which consists of all local affine maps of $\tilde{\nabla}$ preserving its holonomy bundle $\mathcal{P}^{\tilde{\nabla}}$, that is, $\tilde{f}(\mathcal{P}^{\tilde{\nabla}}) \subset \mathcal{P}^{\tilde{\nabla}}$.

This gives \mathcal{G} a structure of Lie pseudo-group. We just have to show that (M, g, \mathcal{G}) is reductive. For a fixed point $p \in M$, the isotropy pseudo-group is

$$\mathcal{H}_p = \{f^\gamma / f^\gamma(p) = p\}$$

which is in one to one correspondence with the set of loops based at p . The linear isotropy group is thus

$$H_p = \{f_*^\gamma : T_p M \rightarrow T_p M / f^\gamma \in \mathcal{H}_p\} = \text{Hol}^{\tilde{\nabla}}.$$

Therefore, let $(\mathfrak{g}, \mathfrak{k})$ be the transitive Lie algebra associated with \mathcal{G} , we have $\mathfrak{k} \simeq \mathfrak{hol}^{\tilde{\nabla}}$. We fix an orthonormal reference u_0 at p and consider the bundle Q defined in (2). Q is exactly the holonomy bundle of $\tilde{\nabla}$ at u_0 , and therefore the connection $\tilde{\nabla}$ reduces to Q and determines a horizontal distribution HQ which is invariant by the right action of H_p and by the left action of \mathcal{G} on Q . We again take the linear map

$$\begin{aligned} \Psi : \quad \mathfrak{g} &\rightarrow T_{u_0} Q \\ [\xi] &\rightarrow \left. \frac{d}{dt} \right|_{t=0} \widetilde{\varphi}_t(u_0), \end{aligned}$$

As seen before Ψ is a linear isomorphism. We consider the subspace $\mathfrak{m} = \Psi^{-1}(H_{u_0} Q) \subset \mathfrak{g}$. Obviously $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$, as $\Psi(\mathfrak{k}) = V_{u_0} Q$. In addition, let $[\xi] \in \mathfrak{m}$ with 1-parameter group φ_t , and let $F = f_* \in H_p$, recall that $\text{Ad}_F([\xi]) = [\eta]$ with $\eta_q = \left. \frac{d}{dt} \right|_{t=0} f \circ \varphi_t \circ f^{-1}(q)$ for every q in a certain neighborhood of p . Hence

$$\Psi(\text{Ad}_F([\xi])) = \left. \frac{d}{dt} \right|_{t=0} \widetilde{f} \circ \varphi_t \circ \widetilde{f}^{-1}(u_0) = \widetilde{f}_* \left((R_{F^{-1}})_* (\widetilde{\xi}_{u_0}) \right).$$

Since $[\xi] \in \mathfrak{m}$ we have that $\Psi([\xi]) \in H_{u_0} Q$, whence by the invariance and the equivariance of the horizontal distribution

$$\widetilde{f}_* \left((R_{F^{-1}})_* (\Psi([\xi])) \right) \in \widetilde{f}_* \left(H_{R_{F^{-1}}(u_0)} Q \right) = H_{u_0} Q.$$

This implies that \mathfrak{m} is $\text{Ad}(H_p)$ -invariant, showing that (M, g) is reductive. \blacksquare

Remark 3.9 *A globally homogeneous pseudo-Riemannian manifold is in particular a locally homogeneous pseudo-Riemannian manifold. Therefore the notion of reductivity that we have defined for locally homogeneous pseudo-Riemannian manifolds must coincide with the well known definition of reductive homogeneous space when we consider a Lie group G as the Lie pseudo-group \mathcal{G} . We show below that this is the case.*

Let (M, g) be a globally homogeneous pseudo-Riemannian manifold with a Lie group G of (global) isometries acting transitively on it. Let H_p be the isotropy group at a point $p \in M$. We denote by \mathfrak{g} and \mathfrak{h} the Lie algebras of G and H respectively. Recall that (M, g, G) is said reductive if $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ for some $\text{Ad}(H_p)$ -invariant subspace $\mathfrak{m} \subset \mathfrak{g}$ (see for instance [6]). We denote by $(\mathfrak{g}', \mathfrak{k}')$ the transitive Lie algebra associated with G seen as a Lie pseudo-group of local isometries, that is, \mathfrak{g}' is the set of germs of local infinitesimal transformations of G . The linear isotropy group as defined in Definition 3.1 is just the image of H_p under the linear

isotropy representation λ (see [6, Ch. X]). We also recall the definition of fundamental vector fields: let $\alpha \in \mathfrak{g}$ we define the vector field α^* on M as

$$\alpha_q^* = \left. \frac{d}{dt} \right|_{t=0} L_{\exp(t\alpha)}(q), \quad q \in M,$$

where L_a denotes the left action of $a \in G$ on M . We consider the following map

$$\begin{aligned} \phi: \mathfrak{g} &\rightarrow \mathfrak{g}' \\ \alpha &\mapsto [\alpha^*]. \end{aligned}$$

Note that ϕ is not a Lie algebra homomorphism since $[\alpha, \beta]^* = -[\alpha^*, \beta^*]$. Nevertheless we show that it is a linear isomorphism. Let $\alpha \in \mathfrak{g}$ be such that $[\alpha^*] = 0$, this means that $\alpha^* = 0$ in a neighborhood of p . In particular $\alpha_p^* = 0$ and $A_{\alpha^*}|_p = 0$, so that $\alpha^* = 0$. This implies $\alpha = 0$, that is, ϕ is injective. On the other hand, let $[\xi] \in \mathfrak{g}'$, we consider the 1-parameter group of ξ , which determines a curve $\varphi_t \subset G$. Taking $\alpha = \left. \frac{d}{dt} \right|_{t=0} \varphi_t$ we have $\phi(\alpha) = [\xi]$. This proves that ϕ is surjective. In addition, let $h \in H_p$ so that $h_* \in \lambda(H_p)$, the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\phi} & \mathfrak{g}' \\ \text{Ad}_h \downarrow & \circlearrowleft & \downarrow \text{Ad}_{h_*} \\ \mathfrak{g} & \xrightarrow{\phi} & \mathfrak{g}' \end{array}$$

In fact, let $\alpha \in \mathfrak{g}$, then $\text{Ad}_{h_*}(\alpha^*) = [\eta]$ with

$$\eta_q = \left. \frac{d}{dt} \right|_{t=0} L_h \circ L_{\exp t\alpha} \circ L_{h^{-1}} = (L_h)_* \left(\alpha_{L_{h^{-1}}(q)}^* \right) = (\text{Ad}_h(\alpha))_q^*.$$

We conclude that via ϕ one can transform reductive complements of $(\mathfrak{g}, \mathfrak{h})$ into reductive complements of $(\mathfrak{g}', \mathfrak{h}')$ and viceversa. This means that the notions of reductivity from both the global and the local points of view coincide.

4 Invariant geometric structures

We now consider a locally homogeneous pseudo-Riemannian manifold (M, g) endowed with a geometric structure given by a tensor field P . Following [5] we give the following definition.

Definition 4.1 *An Ambrose-Singer-Kiričenko connection (or ASK-connection for sort) on (M, g, P) is a linear connection $\tilde{\nabla}$ satisfying*

$$\tilde{\nabla}g = 0, \quad \tilde{\nabla}R = 0, \quad \tilde{\nabla}S = 0, \quad \tilde{\nabla}P = 0.$$

Note that an ASK-connection is in particular an AS-connection. We say that the geometric structure given by P is invariant if the Lie pseudo-group of isometries \mathcal{I} preserving P , that is

$$\mathcal{I} = \{f \in \mathcal{I}, f^*P = P\},$$

acts transitively on M . The corresponding Lie equation is

$$\mathcal{L}_X P,$$

so that the infinitesimal transformations of \mathcal{G} are Killing vector fields which are infinitesimal automorphisms of the geometric structure. A vector field ξ satisfying both $\mathcal{L}_\xi g = 0$ and $\mathcal{L}_\xi P = 0$ will be called a *geometric Killing vector field*. We consider the Lie algebra $\mathfrak{j} \subset \mathfrak{i}$, which consists of germs of geometric Killing vector fields. The Lie subalgebra $\mathfrak{j}^0 \subset \mathfrak{i}^0$ is defined as the set of elements of \mathfrak{j} vanishing at p , so that $(\mathfrak{j}, \mathfrak{j}^0)$ is a transitive Lie algebra. Let \mathfrak{gkill} be the subalgebra of \mathfrak{kil} formed by all Killing generators (X, A) satisfying

$$A \cdot \nabla^j P_p + i_X \nabla^{j+1} P_p = 0, \quad j \geq 0,$$

and let $\mathfrak{gkill}^0 = \mathfrak{kil}^0 \cap \mathfrak{gkill}$, we have

Proposition 4.2 *The transitive Lie algebra $(\mathfrak{gkill}, \mathfrak{gkill}^0)$ is isomorphic to $(\mathfrak{j}, \mathfrak{j}^0)$.*

Proof. Let ξ be a geometric Killing vector field, let $(X, A) = (\xi_p, A_\xi|_p)$. By definition we have

$$A \cdot \nabla^j P = \mathcal{L}_\xi(\nabla^j P)_p - \nabla_\xi \nabla^j P_p = \mathcal{L}_\xi(\nabla^j P)_p - i_X \nabla^{j+1} P_p,$$

and applying Lemma 4.3 below we obtain that $(\xi_p, A_\xi|_p) \in \mathfrak{gkill}$. Making use of Lemma 2.2 and Corollary 2.3 we see that the map

$$\begin{aligned} \mathfrak{j} &\rightarrow \mathfrak{gkill} \\ [\xi] &\mapsto (\xi_p, A_\xi|_p) \end{aligned}$$

is a Lie algebra isomorphism taking \mathfrak{j}^0 to \mathfrak{gkill}^0 . ■

Lemma 4.3 *Let ξ be a Killing vector field and ω a tensor field. If $\mathcal{L}_\xi \omega = 0$ then $\mathcal{L}_\xi(\nabla \omega) = 0$.*

Proof. For the sake of simplicity we show the proof for ω a 1-form. The generalization for tensor fields of arbitrary type is straightforward. By direct calculation

$$\mathcal{L}_\xi(\nabla \omega)(X, Y) = -\xi \cdot (\omega(\nabla_X Y)) + \omega(\nabla_{\mathcal{L}_\xi X} Y) + \omega(\nabla_X \mathcal{L}_\xi Y).$$

Making use of $\mathcal{L}_\xi \omega = 0$ we obtain

$$\mathcal{L}_\xi(\nabla \omega)(X, Y) = \omega((\mathcal{L}_\xi \nabla)(X, Y)) = \omega(R_{\xi X} Y + \nabla_{X Y}^2 \xi).$$

But $R_\xi + \nabla^2 \xi = 0$ since it is just the affine Jacobi equation applied to a Killing vector field ξ . ■

We now consider a Lie pseudo-group $\mathcal{G} \subset \mathcal{J}$ acting transitively on M . We associate to \mathcal{G} the Lie algebra $\mathfrak{g} \subset \mathfrak{j}$ consisting on germs of local geometric Killing vector fields with 1-parameter group contained in \mathcal{G} . The Lie algebra \mathfrak{k} formed by those $[\xi] \in \mathfrak{g}$ vanishing at p is thus a Lie subalgebra of \mathfrak{j}^0 , and the pair $(\mathfrak{g}, \mathfrak{k})$ is a transitive Lie algebra. We take the isotropy pseudo-group \mathcal{H}_p and the linear isotropy group H_p associated with \mathcal{G} . As before we have that H_p is a Lie subgroup of the stabilizer of P_p in $O(T_p M)$, and that $\mathfrak{k} \simeq \mathfrak{h}_p$. Recall also that we have the action Ad of H_p on \mathfrak{g} .

Definition 4.4 Let (M, g, P) be a pseudo-Riemannian manifold endowed with a geometric structure defined by a tensor field P . Let \mathcal{G} be a Lie pseudo-group of isometries acting transitively on (M, g, P) and preserving P . We will say that (M, g, P, \mathcal{G}) is reductive if the transitive Lie algebra $(\mathfrak{g}, \mathfrak{k})$ associated with \mathcal{G} can be decomposed as $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$, where \mathfrak{m} is $\text{Ad}(H_p)$ -invariant.

Theorem 4.5 Let (M, g, P, \mathcal{G}) be a reductive locally homogeneous pseudo-Riemannian manifold with P invariant. Then (M, g, P) admits an ASK-connection.

Proof. Let (M, g, P, \mathcal{G}) be a reductive locally homogeneous pseudo-Riemannian manifold with P invariant, by Theorem 3.7 (M, g) admits an AS-connection $\tilde{\nabla}$. We just have to show that $\tilde{\nabla}P = 0$. However, recall that $\tilde{\nabla}$ is characterized as the linear connection whose parallel transport coincides with the differential h_* of some $h \in \mathcal{G}$. Since \mathcal{G} preserves P , we have that P is invariant by parallel transport with respect to $\tilde{\nabla}$, whence $\tilde{\nabla}P = 0$. ■

Theorem 4.6 Let (M, g, P) be a pseudo-Riemannian manifold admitting an ASK-connection $\tilde{\nabla}$. Then there is a Lie pseudo-group of isometries \mathcal{G} acting transitively on (M, g, P) and preserving P , such that (M, g, P, \mathcal{G}) is reductive locally homogeneous with P invariant.

Proof. As in the proof of Theorem 3.8 we consider the Lie pseudo-group

$$\mathcal{G} = \{f^\gamma / \gamma \text{ is a path from } p \text{ to } q\}.$$

Since the local maps f^γ are affine maps of $\tilde{\nabla}$, and $\tilde{\nabla}P = 0$, we have that P is invariant by \mathcal{G} . The same exact arguments used in the proof of Theorem 3.8 show that (M, g, P) is reductive locally homogeneous with P invariant. ■

5 Strongly reductive locally homogeneous pseudo-Riemannian manifolds

The results presented in this section apply to pseudo-Riemannian metrics of any signature (including the Riemannian case) with or without an extra geometric structure. In addition, these results are new for pseudo-Riemannian metrics with or without an extra geometric structure excluding the case of definite metrics, and in the Riemannian case the results are new in the presence of a geometric structure. For the already known case of Riemannian metrics without extra geometry see [10]. For the sake of brevity we present here the most general case.

Let (M, g) be a pseudo-Riemannian manifold endowed with a geometric structure defined by a tensor field P . Let $p \in M$, for every integers $r, s \geq 0$ we consider the Lie algebras $\mathfrak{g}(p, r)$ and $\mathfrak{p}(p, s)$ given by

$$\mathfrak{g}(p, r) = \left\{ A \in \mathfrak{so}(T_p M), A \cdot \left(\nabla^i R_p \right) = 0, \quad i = 0, \dots, r \right\},$$

$$\mathfrak{p}(p, s) = \left\{ A \in \mathfrak{so}(T_p M), A \cdot \left(\nabla^j P_p \right) = 0, \quad j = 0, \dots, s \right\},$$

where A acts as a derivation on the tensor algebra of $T_p M$. We thus have filtrations

$$\mathfrak{so}(T_p M) \supset \mathfrak{g}(p, 0) \supset \dots \supset \mathfrak{g}(p, r) \supset \dots$$

$$\mathfrak{so}(T_p M) \supset \mathfrak{p}(p, 0) \supset \dots \supset \mathfrak{p}(p, s) \supset \dots$$

Let $k(p)$ and $l(p)$ be the first integers such that $\mathfrak{g}(p, k(p)) = \mathfrak{g}(p, k(p) + 1)$ and $\mathfrak{p}(p, l(p)) = \mathfrak{p}(p, l(p) + 1)$, and let $\mathfrak{h}(p, r, s) = \mathfrak{g}(p, r) \cap \mathfrak{p}(p, s)$. We consider the complex of filtrations

$$\begin{array}{ccccccc} \mathfrak{so}(T_p M) & \supset & \mathfrak{g}(p, 0) & \supset & \dots & \supset & \mathfrak{g}(p, k(p)) = \mathfrak{g}(p, k(p) + 1) \\ \cup & & \cup & & & & \cup \\ \mathfrak{p}(p, 0) & \supset & \mathfrak{h}(p, 0, 0) & \supset & \dots & \supset & \mathfrak{h}(p, k(p), 0) = \mathfrak{h}(p, k(p) + 1, 0) \\ \cup & & \cup & & & & \cup \\ \vdots & & \vdots & & & & \vdots \\ \cup & & \cup & & & & \cup \\ \mathfrak{p}(p, l(p)) & \supset & \mathfrak{h}(p, 0, l(p)) & \supset & \dots & \supset & \mathfrak{h}(p, k(p), l(p)) = \mathfrak{h}(p, k(p) + 1, l(p)) \\ \parallel & & \parallel & & & & \parallel \\ \mathfrak{p}(p, l(p) + 1) & \supset & \mathfrak{h}(p, 0, l(p) + 1) & \supset & \dots & \supset & \mathfrak{h}(p, k(p), l(p) + 1) = \mathfrak{h}(p, k(p) + 1, l(p) + 1). \end{array}$$

To complete the notation we will denote $\mathfrak{g}(p, -1) = \mathfrak{so}(T_p M)$, $\mathfrak{p}(p, -1) = \mathfrak{so}(T_p M)$, so that $\mathfrak{h}(p, -1, s) = \mathfrak{p}(p, s)$ and $\mathfrak{h}(p, r, -1) = \mathfrak{g}(p, r)$.

We shall call a pair of integers $(r(p), s(p))$ in the set $\mathbb{N} \cup \{0, -1\}$ a *stabilizing pair* at $p \in M$ if $r(p) \leq k(p)$, $s(p) \leq l(p)$ and

$$\begin{array}{ccc} \mathfrak{h}(p, r(p), s(p)) & = & \mathfrak{h}(p, r(p) + 1, s(p)) \\ \parallel & & \parallel \\ \mathfrak{h}(p, r(p), s(p) + 1) & = & \mathfrak{h}(p, r(p) + 1, s(p) + 1). \end{array}$$

Note that $(k(p), l(p))$ is a stabilizing pair.

Remark 5.1 An example of a manifold with an stabilizing pair distinct from $(k(p), l(p))$ is exhibited in Section 7.

The following definition generalizes the definition of infinitesimal homogeneous space given by Singer (see [10]). Consider a pair of integers $(r, s) \in (\mathbb{N} \cup \{0, -1\})^2$. We say that (M, g, P) is (r, s) -*infinitesimally P-homogeneous* if for every $p, q \in M$ there is a linear isometry $F : T_p M \rightarrow T_q M$ such that

$$F^*(\nabla^i R_q) = \nabla^i R_p, \quad i = 0, \dots, r + 1,$$

$$F^*(\nabla^j P_q) = \nabla^j P_p, \quad j = 0, \dots, s + 1.$$

Let $p \in M$ be a fixed point and suppose that $(r(p), s(p))$ is a stabilizing pair at p . If (M, g, P) is $(r(p), s(p))$ -infinitesimally P -homogeneous, then $(r(p), s(p))$ is a stabilizing pair at all $q \in M$ (so that we can omit the point p). In fact, any linear isometry $F : T_p M \rightarrow T_q M$ with $F^*(\nabla^i R_q) = \nabla^i R_p$ and $F^*(\nabla^j P_q) = \nabla^j P_p$ for $i = 0, \dots, r(p) + 1$ and $j = 0, \dots, s(p) + 1$, induces isomorphisms between $\mathfrak{h}(p, i, j)$ and $\mathfrak{h}(q, i, j)$ for $i \leq r(p)$ and $j \leq s(p)$. Note that this means that if (M, g, P) is $(k(p), l(p))$ -infinitesimally P -homogeneous then the numbers $k(q)$ and $l(q)$ are independent of $q \in M$. Let $H(p, r, s)$ be the stabilizing group of the tensors $\nabla^i R_p$, and $\nabla^j P_p$,

$0 \leq i \leq r+1, 0 \leq j \leq s+1$, inside $O(T_p M)$. It is evident that $\mathfrak{h}(p, r, s)$ is the Lie algebra of $H(p, r, s)$.

Obviously, a locally homogeneous pseudo-Riemannian manifold with P invariant is in particular (r, s) -infinitesimally P -homogeneous for every pair (r, s) . We shall see that the converse is also true.

Definition 5.2 *Let (r, s) be a stabilizing pair at $p \in M$. (M, g, P) is said (r, s) -strongly reductive at p if there is an $\text{Ad}(H(p, r, s))$ -invariant subspace $\mathfrak{n}(p, r, s) \subset \mathfrak{so}(T_p M)$ such that*

$$\mathfrak{so}(T_p M) = \mathfrak{h}(p, r, s) \oplus \mathfrak{n}(p, r, s).$$

Lemma 5.3 *Let (M, g, P) be (r, s) -infinitesimally P -homogeneous. If (M, g, P) is (r, s) -strongly reductive at $p \in M$, then it is (r, s) -strongly reductive at every point of M .*

Proof. Let $q \in M$ be another point distinct from p , recall that (r, s) is also a stabilizing pair at q . Let $F : T_p M \rightarrow T_q M$ be a linear isometry such that $F^*(\nabla^i R_q) = \nabla^i R_p$ and $F^*(\nabla^j P_q) = \nabla^j P_p$ for $i = 0, \dots, r+1$ and $j = 0, \dots, s+1$. F induces a linear isomorphism $\tilde{F} : \mathfrak{so}(T_p M) \rightarrow \mathfrak{so}(T_q M)$ given by $A \mapsto F \circ A \circ F^{-1}$. By construction it is obvious that $\tilde{F}(\mathfrak{h}(p, r, s)) = \mathfrak{h}(q, r, s)$. Let $\mathfrak{n}(p, r, s)$ be an $\text{Ad}(H(p, r, s))$ -invariant complement to $\mathfrak{h}(p, r, s)$ inside $\mathfrak{so}(T_p M)$, we define

$$\mathfrak{n}(q, r, s) = \tilde{F}(\mathfrak{n}(p, r, s)) \subset \mathfrak{so}(T_q M).$$

This subspace is independent of the isometry F . Indeed, let $G : T_p M \rightarrow T_q M$ be another linear isometry with $G^*(\nabla^i R_q) = \nabla^i R_p$ and $G^*(\nabla^j P_q) = \nabla^j P_p$ for $i = 0, \dots, r+1$ and $j = 0, \dots, s+1$. The composition $G^{-1} \circ F$ is an element of $O(T_p M)$. Moreover, $G^{-1} \circ F$ stabilizes $R_p, \dots, \nabla^{r+1} R_p$ and $P_p, \dots, \nabla^{s+1} P_p$, so that it is an element of $H(p, r, s)$. Hence, for any $A \in \mathfrak{n}(p, r, s)$ we have

$$\tilde{G}^{-1} \circ \tilde{F}(A) = \text{Ad}_{G^{-1} \circ F}(A) \in \mathfrak{n}(p, r, s),$$

showing that $\tilde{F}(\mathfrak{n}(p, r, s))$ does not depend on the linear isometry F . We finally show that $\mathfrak{n}(q, r, s)$ is $\text{Ad}(H(q, r, s))$ -invariant. Let $B \in \mathfrak{n}(q, r, s)$, there exists an element $A \in \mathfrak{n}(p, r, s)$ with $B = \tilde{F}(A)$. Let $b \in H(q, r, s)$, we take $a = F^{-1} \circ b \circ F \in H(p, r, s)$. Then

$$\text{Ad}_b(B) = b \circ B \circ b^{-1} = F \circ a \circ A \circ a^{-1} \circ F^{-1} = \tilde{F}(\text{Ad}_a(A)),$$

which belongs to $\mathfrak{n}(q, r, s)$ since $\text{Ad}_a(A) \in \mathfrak{n}(p, r, s)$. ■

By virtue of the previous Lemma, we say that an (r, s) -infinitesimally P -homogeneous manifold (M, g, P) is (r, s) -strongly reductive if it is (r, s) -strongly reductive at some point of M . The same applies for locally homogeneous spaces with P invariant. The term “strongly reductive” is motivated by Proposition 5.11 and Example 7.3, which show that strong reductivity implies reductivity, but the converse is not true.

Remark 5.4 In the case g is Riemannian, the Killing form of $\mathfrak{so}(T_p M)$ is definite, so that the strong reductivity condition is automatically satisfied choosing for $\mathfrak{n}(p, r, s)$ the orthogonal complement of $\mathfrak{h}(p, r, s)$ inside $\mathfrak{so}(T_p M)$ with respect to the Killing form. When the presence of an extra geometric structure is not taken into account, the integer $k(p)$ stabilizing the filtration

$$\mathfrak{so}(T_p M) \supset \mathfrak{g}(p, 0) \supset \dots \supset \mathfrak{g}(p, r) \supset \dots$$

is known as the Singer invariant of (M, g) . In this case, the choice of $\mathfrak{g}(p, k(p))^\perp$ as complement of $\mathfrak{g}(p, k(p))$ leads to the canonical AS-connection constructed by Kowalski in [8] in a similar way to the proof of Theorem 5.8 below.

Let $\pi : \mathcal{O}(M) \rightarrow M$ be the bundle of orthonormal references with structure group $O(\nu, n - \nu)$, where ν is the index of the metric. Let $u_0 \in \mathcal{O}(M)$ with $\pi(u_0) = p$, and $P_0 = u_0^*(P_p)$. Let \mathbf{P} be the space of tensors to which P_0 belongs. For any pair of integers $(r, s) \in (\mathbb{N} \cup \{0, -1\})^2$ we consider the following $O(\nu, n - \nu)$ -equivariant map:

$$\begin{aligned} \Phi_{(r,s)} : \mathcal{O}(M) &\rightarrow \bigoplus_{i=0}^{k+1} (\bigotimes^{r+4} (\mathbb{R}^n)^*) \oplus \bigoplus_{j=0}^{s+1} (\bigotimes^j (\mathbb{R}^n)^* \otimes \mathbf{P}) \\ u &\mapsto u^*(R_{\pi(u)}, \dots, \nabla^{r+1} R_{\pi(u)}, P_{\pi(u)}, \dots, \nabla^{s+1} P_{\pi(u)}). \end{aligned}$$

Lemma 5.5 If (M, g, P) is (r, s) -infinitesimally P -homogeneous, then the image of $\mathcal{O}(M)$ under $\Phi_{(r,s)}$ is a single $O(\nu, n - \nu)$ -orbit.

Proof. Let $u \in \mathcal{O}(M)$ and denote $\Phi = \Phi_{(r,s)}$. If $\pi(u_0) = \pi(u)$ then u_0 and u are in the same $O(\nu, n - \nu)$ -orbit, and since Φ is $O(\nu, n - \nu)$ -equivariant, we have that $\Phi(u_0)$ and $\Phi(u)$ are in the same $O(\nu, n - \nu)$ -orbit. If $\pi(u_0) \neq \pi(u)$, let $q = \pi(u)$, then there is a linear isometry $F : T_p M \rightarrow T_q M$ such that $F^*(\nabla^i R_q) = \nabla^i R_p$ for $i = 0, \dots, r + 1$, and $F^*(\nabla^j P_q) = \nabla^j P_p$ for $j = 0, \dots, s + 1$. F induces a map $\tilde{F} : \mathcal{O}(M) \rightarrow \mathcal{O}(M)$ such that $\Phi \circ \tilde{F} = \Phi$. Since $\pi(u) = \pi(\tilde{F}(u_0))$, we conclude that $\Phi(u_0)$ and $\Phi(u)$ are in the same $O(\nu, n - \nu)$ -orbit. ■

Lemma 5.6 If (M, g, P) is an (r, s) -infinitesimally P -homogeneous manifold. Then there is a metric connection $\bar{\nabla}$ such that $\bar{\nabla}_X(\nabla^i R) = 0$ for $i = 0, \dots, r + 1$, and $\bar{\nabla}_X(\nabla^j P) = 0$ for $j = 0, \dots, s + 1$.

Proof. Let $u_0 \in P$ with $\pi(u_0) = p$ and $\Phi = \Phi_{(r,s)}$. By Lemma 5.5 the orbit $\Phi(P)$ is the homogeneous space $O(\nu, n - \nu)/I_0$ where I_0 is the isotropy group of $\Phi(u_0)$. We thus have an equivariant map $\Phi : \mathcal{O}(M) \rightarrow O(\nu, n - \nu)/I_0$, so that $Q = \Phi^{-1}(\Phi(u_0))$ determines a reduction of $\mathcal{O}(M)$ with group I_0 . Since Φ restricted to Q is constant, all the tensor fields $\nabla^i R$ and $\nabla^j P$, $i = 0, \dots, r + 1$, $j = 0, \dots, s + 1$, will be parallel with respect to any connection adapted to Q . ■

Lemma 5.7 If (M, g, P) is an (r, s) -infinitesimally P -homogeneous manifold, then

$$\mathfrak{h}(M, r, s) = \bigcup_{q \in M} \mathfrak{h}(q, r, s)$$

is a vector subbundle of $\mathfrak{so}(M)$. If (M, g, P) is moreover (r, s) -strongly reductive, then

$$\mathfrak{n}(M, r, s) = \bigcup_{q \in M} \mathfrak{n}(q, r, s)$$

is a vector subbundle of $\mathfrak{so}(M)$ and $\mathfrak{so}(M) = \mathfrak{h}(M, r, s) \oplus \mathfrak{n}(M, r, s)$.

Proof. To prove that $\mathfrak{h}(M, r, s)$ is a vector subbundle of $\mathfrak{so}(M)$ we have to find a neighborhood U around every $q \in M$ admitting local sections $\{H_1, \dots, H_t\}$ such that $\{H_1(y), \dots, H_t(y)\}$ is a basis of $\mathfrak{h}(y, r, s)$ for every $y \in U$. Let $\bar{\nabla}$ be a linear connection as in Lemma 5.6, we take a normal neighborhood U around q with respect to $\bar{\nabla}$. Let $\{H_1(q), \dots, H_t(q)\}$ be a basis of $\mathfrak{h}(q, r, s)$, we extend them by parallel transport with respect to $\bar{\nabla}$ along radial $\bar{\nabla}$ -geodesics in order to define $\{H_1(y), \dots, H_t(y)\}$. Since $\bar{\nabla}_X(\nabla^i R) = 0$ for $i = 0, \dots, r+1$, and $\bar{\nabla}_X(\nabla^j P) = 0$ for $j = 0, \dots, s+1$, the parallel transport from q to y defines a linear isometry $F : T_q M \rightarrow T_y M$ with $F^*(\nabla^i R_y) = \nabla^i R_q$ for $i = 0, \dots, r+1$, and $F^*(\nabla^j P_y) = \nabla^j P_q$ for $j = 0, \dots, s+1$. This implies that $H_i(y) \in \mathfrak{h}(y, r, s)$. If (M, g, P) is (r, s) -strongly reductive, we consider the decomposition $\mathfrak{so}(T_q M) = \mathfrak{h}(q, r, s) \oplus \mathfrak{n}(q, r, s)$ and take a basis $\{\eta_1(q), \dots, \eta_d(q)\}$ of $\mathfrak{n}(q, r, s)$. Extending $\{\eta_1(q), \dots, \eta_d(q)\}$ by parallel transport along radial $\bar{\nabla}$ -geodesics, we obtain local sections η_1, \dots, η_d of $\mathfrak{so}(M)$ defined on U . As seen in Lemma (5.3), the linear isometries F determined by the parallel transport takes $\mathfrak{n}(q, r, s)$ to $\mathfrak{n}(y, r, s)$ for $y \in U$, whence $\{\eta_1(y), \dots, \eta_d(y)\}$ is a basis of $\mathfrak{n}(y, r, s)$ for every $y \in U$. ■

Theorem 5.8 *Let (M, g, P) be an (r, s) -infinitesimally P -homogeneous manifold. If (M, g, P) is (r, s) -strongly reductive with a decomposition $\mathfrak{so}(T_p M) = \mathfrak{n}(p, r, s) \oplus \mathfrak{h}(p, r, s)$ with $\mathfrak{n}(M, r, s)$ $\text{Ad}(H(p, r, s))$ -invariant, then there is a unique ASK-connection $\tilde{\nabla}$ such that $S = \tilde{\nabla} - \nabla$ is a section of $T^*M \otimes \mathfrak{n}(M, r, s)$.*

Proof. Let $\mathfrak{h}(M)$ denote $\mathfrak{h}(M, r, s)$ and let $\mathfrak{n}(M)$ denote $\mathfrak{n}(M, r, s)$. Let $\bar{\nabla}$ be a linear connection as in Lemma 5.6. We consider the tensor field $B = \nabla - \bar{\nabla}$, which defines a section of $T^*M \otimes \mathfrak{so}(M)$ as $\bar{\nabla}$ is metric. In virtue of Lemma 5.7 we decompose

$$B = B^{\mathfrak{h}} + B^{\mathfrak{n}},$$

with $B^{\mathfrak{h}}$ and $B^{\mathfrak{n}}$ sections of $T^*M \otimes \mathfrak{h}(M)$ and $T^*M \otimes \mathfrak{n}(M)$ respectively. We define $S = B^{\mathfrak{n}}$, and take $\tilde{\nabla} = \nabla - S$. Since S is a section of $T^*M \otimes \mathfrak{so}(M)$ we have that $\tilde{\nabla}$ is metric, so that $\tilde{\nabla}g = 0$. Moreover

$$\tilde{\nabla}_X(\nabla^i R) = \bar{\nabla}_X(\nabla^i R) + B_X^{\mathfrak{h}} \cdot (\nabla^i R) = 0, \quad i = 0, \dots, r+1,$$

$$\tilde{\nabla}_X(\nabla^j P) = \bar{\nabla}_X(\nabla^j P) + B_X^{\mathfrak{h}} \cdot (\nabla^j P) = 0, \quad j = 0, \dots, s+1,$$

since (r, s) is a stabilizing pair. Finally, let $q \in M$ and consider a normal neighborhood of q with respect to $\tilde{\nabla}$. Since

$$0 = \tilde{\nabla}_X(\nabla^i R) = i_X(\nabla^{i+1} R) - S_X \cdot (\nabla^i R),$$

$$0 = \tilde{\nabla}_X(\nabla^j P) = i_X(\nabla^{j+1} P) - S_X \cdot (\nabla^j P),$$

differentiating these formulae along a radial $\tilde{\nabla}$ -geodesic $\gamma(t)$ we find

$$0 = 0 - \frac{d}{dt} \left(S_{\dot{\gamma}(t)} \cdot (\nabla^i R)_{\gamma(t)} \right) = - \left(\tilde{\nabla}_{\dot{\gamma}(t)} S \right) \cdot (\nabla^i R)_{\gamma(t)},$$

$$0 = 0 - \frac{d}{dt} \left(S_{\dot{\gamma}(t)} \cdot (\nabla^j P)_{\gamma(t)} \right) = - \left(\tilde{\nabla}_{\dot{\gamma}(t)} S \right) \cdot (\nabla^j P)_{\gamma(t)},$$

for $i = 0, \dots, r$ and $j = 0, \dots, s$. This means that $\tilde{\nabla}_{\dot{\gamma}(t)} S \in \mathfrak{h}(\gamma(t), r, s)$. In addition, as a consequence of the $\text{ad}(\mathfrak{h}(M))$ -invariance of $\mathfrak{n}(M)$, the covariant derivative of a section of $\mathfrak{n}(M)$ is again a section of $\mathfrak{n}(M)$, so that $\tilde{\nabla}_{\dot{\gamma}(t)} S \in \mathfrak{n}(\gamma(t), r, s)$. We conclude that $\tilde{\nabla} S = 0$.

We finally prove uniqueness. Let $\tilde{\nabla}$ and $\tilde{\nabla}'$ be as in the hypothesis, then $S - S'$ is a section of $T^*M \otimes \mathfrak{n}(M)$. In addition $\tilde{\nabla}(\nabla^i R) = \tilde{\nabla}'(\nabla^i R) = 0$ and $\tilde{\nabla}(\nabla^j P) = \tilde{\nabla}'(\nabla^j P) = 0$ for all i, j . These are easily obtained from the fact that the torsion and the curvature of $\tilde{\nabla}$ (resp. $\tilde{\nabla}'$) are parallel with respect to $\tilde{\nabla}$ (resp. $\tilde{\nabla}'$), and from $\tilde{\nabla} P = \tilde{\nabla}' P = 0$. This implies that $S - S'$ is a section of $T^*M \otimes \mathfrak{h}(M)$, and then $S = S'$ and $\tilde{\nabla} = \tilde{\nabla}'$. ■

Corollary 5.9 *Let (r, s) and (r', s') be stabilizing pairs. If $\mathfrak{n}(p, r, s) \subset \mathfrak{n}(p, r', s')$, then the connections $\tilde{\nabla}$ and $\tilde{\nabla}'$ constructed from them coincide.*

Proof. This is evident since $S = \tilde{\nabla} - \nabla$ is a section of both $\mathfrak{n}(M, r, s)$ and $\mathfrak{n}(M, r', s')$. ■

As we have seen, a strongly reductive locally homogeneous pseudo-Riemannian manifold (M, g, P) with P invariant admits an ASK-connection, so by Theorem 3.8 there is a Lie pseudo-group \mathcal{G} (which is not necessarily the full isometry pseudo-group) acting transitively by isometries and preserving P such that (M, g, P, \mathcal{G}) is reductive. Moreover, we shall show that strongly reductive locally homogeneous spaces with an invariant geometric structure P are reductive for the action of the full pseudo-group of isometries preserving P . In order to prove that we will make use of some results contained in Section 6 and the following Lemma.

Lemma 5.10 *Let $\tilde{\nabla}$ be an ASK-connection with curvature K and torsion T . Let $p \in M$, and let $A \in \mathfrak{so}(T_p M)$ be such that $A \cdot K_p = 0$, $A \cdot T_p = 0$ and $A \cdot P_p = 0$. Then $A \cdot \nabla^i R_p = 0$ and $A \cdot \nabla^j P_p = 0$ for all $i, j \geq 0$.*

Proof. The curvature and torsion of $\tilde{\nabla}$ are related to R and S by

$$T_X Y = S_Y X - S_X Y, \quad K_{XY} = R_{XY} + [S_X, S_Y] + S_{T_X Y}.$$

Making use of these formulae in conjunction with $\tilde{\nabla} R = 0$ and $\tilde{\nabla} S = 0$, an inductive argument gives that $\tilde{\nabla}(\nabla^i R) = 0$ for all $i \geq 0$. A similar computation gives $\tilde{\nabla}(\nabla^j P) = 0$ for all $j \geq 0$. This means that

$$i_X \nabla^{i+1} R = S_X \cdot \nabla^i R, \quad i_X \nabla^{j+1} P = S_X \cdot \nabla^j P,$$

for all $i, j \geq 0$. Let now $A \in \mathfrak{so}(T_p M)$ be such that $A \cdot K_p = 0$, $A \cdot T_p = 0$ and $A \cdot P_p = 0$. By Corollary 6.7 $A \cdot S_p = 0$, hence $A \cdot R_p = 0$. A simple computation making use of the previous formulae leads to

$$(A \cdot \nabla^{i+1} R_p)_X = (A \cdot S_p)_X \cdot \nabla^i R_p + (S_p)_X \cdot (A \cdot \nabla^i R_p), \quad i \geq 0,$$

$$(A \cdot \nabla^{j+1} P_p)_X = (A \cdot S_p)_X \cdot \nabla^j P_p + (S_p)_X \cdot (A \cdot \nabla^j P_p), \quad j \geq 0.$$

Therefore, by induction on i and j we obtain that $A \cdot \nabla^i R_p = 0$ and $A \cdot \nabla^j P_p = 0$ for all $i, j \geq 0$. ■

Proposition 5.11 *If (M, g, P) is (r, s) -strongly reductive, then (M, g, \mathcal{J}) is reductive, where \mathcal{J} is the full Lie pseudo-group of local isometries preserving P .*

Proof. Let $\mathfrak{so}(T_p M) = \mathfrak{n}(p, r, s) \oplus \mathfrak{h}(p, r, s)$, and let $\tilde{\nabla}$ be the associated ASK-connection. Let K and T be the curvature and the torsion tensor fields of $\tilde{\nabla}$ respectively. The triple (K, T, P) defines an infinitesimal model (see Definition 6.4 and Proposition 6.8), and we can consider the associated Nomizu construction, that is, we define the Lie algebra $\mathfrak{g}_0 = T_p M \oplus \mathfrak{h}_0$ with the usual brackets, where

$$\mathfrak{h}_0 = \{A \in \mathfrak{so}(T_p M) \mid A \cdot K_p = 0, A \cdot T_p = 0, A \cdot P_p = 0\}.$$

By Proposition 6.9 the Lie algebra \mathfrak{h}_0 is equal to $\mathfrak{h}(p, r, s)$. On the other hand, $\mathfrak{h}_0 \subset \mathfrak{g}^{\text{fill}}_0 \simeq \mathfrak{j}^0$ by Lemma 5.10, and $\mathfrak{g}^{\text{fill}}_0 \subset \mathfrak{h}$ by definition, whence $\mathfrak{g}^{\text{fill}}_0 \subset \mathfrak{h} = \mathfrak{h}_0$. We thus define the following Lie algebra isomorphism

$$\begin{aligned} \Phi : \quad \mathfrak{g}_0 &\rightarrow \mathfrak{g}^{\text{fill}} \\ X + A &\mapsto (X, (S_0)_X + A). \end{aligned}$$

The image of $T_p M$ defines a complement \mathfrak{m} of $\mathfrak{g}^{\text{fill}}_0$. Making use of Lemma 6.6 we have that $\text{Ad}_B(S_X) = S_{BX}$ for all B in $H(p, r, s)$ and all $X \in T_p M$. Since the linear isotropy group H_p is contained in $H(p, r, s)$ we have that \mathfrak{m} is $\text{Ad}(H_p)$ -invariant. ■

6 Reconstruction of strongly reductive locally homogeneous spaces

We first show a uniqueness result satisfied by strongly reductive locally homogeneous pseudo-Riemannian manifolds with an invariant geometric structure.

Proposition 6.1 *Let (M, g, P) and (M', g', P') be pseudo-Riemannian manifolds with tensor fields P and P' . Suppose (M', g', P') is locally homogeneous with P' invariant. Suppose furthermore that (M', g', P') is (r, s) -strongly reductive for some stabilizing pair (r, s) . If for each point $p \in M$ there is a linear isometry $F : T_p M \rightarrow T_o M'$ (where $o \in M'$ can be fixed) such that $F^*(\nabla'^i R'_o) = \nabla^i R_p$ for $i = 0, \dots, r+1$, and $F^*(\nabla'^j P'_o) = \nabla^j P_p$ for $j = 0, \dots, s+1$. Then (M, g, P) is locally homogeneous with P invariant and locally isometric to (M', g', P') preserving P and P' .*

Proof. Note first of all that (M, g, P) is (r, s) -infinitesimally P -homogeneous and (r, s) -strongly reductive, so that (M, g, P) is locally homogeneous with P invariant. Let $\tilde{\nabla}$ and $\tilde{\nabla}'$ be connections on M and M' respectively as in Theorem 5.8. Let $S = \nabla - \tilde{\nabla}$ and $S' = \nabla' - \tilde{\nabla}'$, and let $F : T_p M \rightarrow T_o M'$ be as in the hypothesis. It is obvious that $F^*(S'_o) - S_p \in T_p^* M \otimes \mathfrak{n}(p, r, s)$. In addition

$$\begin{aligned} (F^*(S'_o)_X - (S_p)_X) \cdot (\nabla^i R_p) &= i_X \nabla^{i+1} R_p - i_X \nabla^{i+1} R_p = 0, \quad i = 0, \dots, r, \\ (F^*(S'_o)_X - (S_p)_X) \cdot (\nabla^j P_p) &= i_X \nabla^{j+1} P_p - i_X \nabla^{j+1} P_p = 0, \quad j = 0, \dots, s, \end{aligned}$$

so that $F^*(S'_o)_X - (S_p)_X \in \mathfrak{h}(p, r, s)$. We conclude that $F^*(S'_o) = S_p$. Since the torsion of $\tilde{\nabla}$ is $S_Y X - S_X Y$, and a similar formula holds for the torsion of $\tilde{\nabla}'$, as a simple inspection shows, F preserves the curvature and the torsion of $\tilde{\nabla}$ and $\tilde{\nabla}'$, which are parallel with respect to $\tilde{\nabla}$. Therefore, there are neighborhoods U and V around p and o respectively, and an affine map $f : U \rightarrow V$ with respect to $\tilde{\nabla}$ and $\tilde{\nabla}'$ (see [6, Ch. 7]). Since $\tilde{\nabla}$ and $\tilde{\nabla}'$ are metric and $\tilde{\nabla}P = \tilde{\nabla}'P' = 0$, we have that f is an isometry preserving P and P' . ■

Theorem 5.8 and Proposition 6.1 suggest the possibility of reconstructing a strongly reductive locally homogeneous manifold (M, g, P) with P invariant from the knowledge of the curvature tensor field, the tensor field P , and their covariant derivatives at a point $p \in M$ up to finite order. In order to prove this result we must first examine the algebraic properties of the curvature tensor field, P and its covariant derivatives.

Let (M, g, P) be a locally homogeneous pseudo-Riemannian manifold with P invariant. We fix a point $p \in M$ and set $V = T_p M$. Consider the tensors $R^i = \nabla^i R_p$ and $P^j = \nabla^j P_p$ for $i, j \geq 0$. One has

$$R_{XYZW}^0 = -R_{YXZW}^0 = R_{ZWXY}^0, \quad (3)$$

$$\bigotimes_{XYZ} R_{XYZW}^0 = 0, \quad (4)$$

$$R_{XYZVW}^1 = -R_{XZYVW}^1 = R_{XVWYZ}^1, \quad (5)$$

$$\bigotimes_{YZV} R_{XYZVW}^1 = 0, \quad (6)$$

$$\bigotimes_{XYZ} R_{XYZVW}^1 = 0, \quad (7)$$

$$R_{YX}^{i+2} - R_{XY}^{i+2} = R_{XY}^0 \cdot R^i, \quad (8)$$

$$P_{YX}^{j+2} - P_{XY}^{j+2} = R_{XY}^0 \cdot P^j, \quad (9)$$

for $i, j \geq 0$, where R_{XY}^0 is acting as a derivation on the tensor algebra. In addition, let $\tilde{\nabla}$ be an ASK-connection and $S = \nabla - \tilde{\nabla}$, we have that

$$i_X R^{i+1} = S_X \cdot R^i, \quad i_X P^{j+1} = S_X \cdot P^j,$$

for $0 \leq i \leq r+1, 0 \leq j \leq s+1$, where (r, s) is a stabilizing pair. We thus consider the following linear maps

$$\begin{aligned} \mu_{i,j} : \mathfrak{so}(V) &\rightarrow W_{i,j} \\ A &\mapsto (A \cdot R^0, \dots, A \cdot R^i, A \cdot P^0, \dots, A \cdot P^j), \end{aligned}$$

and

$$\begin{aligned} \nu : V &\rightarrow W_{r+1,s+1} \\ X &\mapsto (i_X R^1, \dots, i_X R^{r+2}, i_X P^1, \dots, i_X P^{s+2}), \end{aligned}$$

with

$$W_{i,j} = \left[\bigoplus_{\alpha=0}^i (\otimes^{\alpha+4} V^*) \right] \otimes \left[\bigoplus_{\beta=0}^j ((\otimes^{\beta} V^*) \otimes \mathbf{P}) \right],$$

where \mathbf{P} is the space of tensors to which P^0 belongs. The previous discussion for a stabilizing pair (r, s) thus gives

$$\nu(V) \subset \mu_{r+1, s+1}(\mathfrak{so}(V)), \quad (10)$$

and

$$\ker(\mu_{r, s}) = \ker(\mu_{r+1, s}) = \ker(\mu_{r, s+1}) = \ker(\mu_{r+1, s+1}). \quad (11)$$

Finally, let $H(r, s)$ be the stabilizer of R^0, \dots, R^{r+1} and P^0, \dots, P^{s+1} inside $O(V)$. In view of Theorem 5.8, to assure the existence of an ASK-connection we need that

$$\mathfrak{so}(V) = \ker(\mu_{r, s}) \oplus \mathfrak{n} \quad (12)$$

for an $\text{Ad}_{H(r, s)}$ -invariant subspace \mathfrak{n} . We shall prove the following result.

Theorem 6.2 *Let V be a vector space endowed with an inner product \langle, \rangle . Let $R^0, \dots, R^{r+2}, P^0, \dots, P^{s+2}$ be tensors on V satisfying (3), ..., (9) for $0 \leq i \leq r$ and $0 \leq j \leq s$, and such that (10), (11), and (12) hold. Then*

1. *There is an (r, s) -strongly reductive locally homogeneous pseudo-Riemannian manifold (M, g, P) with P invariant, whose curvature tensor field, P , and their covariant derivatives coincide with $R^0, \dots, R^{r+2}, P^0, \dots, P^{s+2}$ at a point $p \in M$. Moreover, (M, g, P) is unique up to local isometry preserving P .*
2. *If the infinitesimal data $R^0, \dots, R^{r+2}, P^0, \dots, P^{s+2}$ is regular (see Definitions 6.5 and 6.10), then there is an (r, s) -strongly reductive globally homogeneous pseudo-Riemannian space $(G_0/H_0, g, P)$, whose curvature tensor field, P , and their covariant derivatives coincide with $R^0, \dots, R^{r+2}, P^0, \dots, P^{s+2}$ at a point $p \in M$. $(G_0/H_0, g, P)$ is moreover unique up to local isometry preserving P .*

Corollary 6.3 *An (r, s) -strongly reductive locally homogeneous pseudo-Riemannian manifold (M, g, P) with P invariant can be reconstructed (up to local isometry) from the data $R_p, \dots, \nabla^{r+2} R_p, P_p, \dots, \nabla^{s+2} P_p$, where (r, s) is a stabilizing pair.*

Before proving Theorem 6.2 we need to recall the definition of *infinitesimal model* and show that an infinitesimal model can be associated to every suitable infinitesimal data $R^0, \dots, R^{s+2}, P^0, \dots, P^{r+2}$ satisfying the hypotheses of Theorem 6.2.

Let V be a vector space with an inner product \langle, \rangle , and let P be a tensor on V . We consider morphisms

$$T : V \rightarrow \text{End}(V), \quad K : V \times V \rightarrow \text{End}(V).$$

Definition 6.4 A triple (T, K, P) is called an infinitesimal model if the following properties are satisfied:

$$T_X Y + T_Y X = 0 \quad (13)$$

$$K_{XY} Z + K_{YX} Z = 0 \quad (14)$$

$$\langle K_{XY} Z, W \rangle + \langle K_{WZ} X, Y \rangle = 0 \quad (15)$$

$$K_{XY} \cdot T = 0 \quad (16)$$

$$K_{XY} \cdot K = 0 \quad (17)$$

$$K_{XY} \cdot P = 0 \quad (18)$$

$$\bigcirc_{XYZ} (K_{XY} Z + T_{TX} Y Z) = 0 \quad (19)$$

$$\bigcirc_{XYZ} K_{T_X Y Z} = 0 \quad (20)$$

When the geometric structure P is absent, an infinitesimal model is just a pair (T, K) satisfying the previous properties with the exception of (18).

Let $\tilde{\nabla}$ be an ASK-connection on (M, g, P) . For a fixed point $p \in M$ we take $V = T_p M$, $\langle \cdot, \cdot \rangle = g_p$, $P = P_p$, and T and K the torsion and curvature of $\tilde{\nabla}$ at p respectively. It is easy to see that in that case (T, K, P) satisfies (13), ..., (20), so that it defines an infinitesimal model. The converse is true under suitable conditions that must be explained. To every infinitesimal model (T, K, P) one can associate the so called *Nomizu construction*, that is, the Lie algebra

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus V,$$

where

$$\mathfrak{h}_0 = \{A \in \mathfrak{so}(V) / A \cdot K = 0, A \cdot T = 0, A \cdot P = 0\},$$

and the Lie brackets are defined by

$$\begin{aligned} [A, B] &= AB - BA, & A, B &\in \mathfrak{h}_0, \\ [A, X] &= A \cdot X, & A &\in \mathfrak{h}_0, X \in V, \\ [X, Y] &= -T_X Y + K_{XY}, & X, Y &\in V. \end{aligned}$$

Note that $K_{XY} \in \mathfrak{h}_0$. Let \mathfrak{h}'_0 be the subalgebra spanned by all elements K_{XY} (which in the case when (T, K, P) comes from an ASK-connection coincides with the holonomy algebra of $\tilde{\nabla}$), the Lie algebra $\mathfrak{g}'_0 = \mathfrak{h}'_0 \oplus V$ is the so called *transvection algebra* (see [7]).

We now consider the abstract simply-connected Lie group G_0 with Lie algebra \mathfrak{g}_0 , and its connected Lie subgroup H_0 with Lie algebra \mathfrak{h}_0 . We also consider the simply-connected Lie group G'_0 with Lie algebra \mathfrak{g}'_0 , and its connected Lie subgroup H'_0 with Lie algebra \mathfrak{h}'_0 .

Definition 6.5 We say that the infinitesimal model (T, K, P) is regular if H_0 is closed in G_0 . On the other hand, we say that the transvection algebra $(\mathfrak{g}'_0, \mathfrak{h}'_0)$ is regular if H'_0 is closed in G'_0 .

In the case when (T, K, P) (resp. the transvection algebra) is regular, the quotient G_0/H_0 (resp. G'_0/H'_0) is a pseudo-Riemannian homogeneous space with an invariant tensor field \bar{P} coinciding with P at the origin.

We now show how to associate an infinitesimal model to every suitable data $R^0, \dots, R^{r+2}, P^0, \dots, P^{s+2}$ on V satisfying the hypotheses of Theorem 6.2. We define $\mathfrak{h} = \ker(\mu_{r+1, s+1})$, and consider an $\text{Ad}(H(r, s))$ -invariant complement \mathfrak{n} of \mathfrak{h} inside $\mathfrak{so}(V)$. From (10) we have that for every $X \in V$ there is an endomorphism $A(X) \in \mathfrak{so}(V)$ such that

$$\begin{aligned} i_X R^{i+1} &= A(X) \cdot R^i, & 0 \leq i \leq r+1, \\ i_X P^{j+1} &= A(X) \cdot P^j, & 0 \leq j \leq s+1. \end{aligned}$$

We decompose $A(X) = A_1(X) + A_2(X)$, where $A_1(X) \in \mathfrak{h}$ and $A_2(X) \in \mathfrak{n}$. Note that $A(X)$ is uniquely determined up to an \mathfrak{h} -component, so that we can take the uniquely defined map

$$\begin{aligned} S: V &\rightarrow \mathfrak{n} \\ X &\mapsto S_X = A_2(X). \end{aligned}$$

By the definition of \mathfrak{h} it is evident that

$$i_X R^{i+1} = S_X \cdot R^i, \quad 0 \leq i \leq r+1, \quad (21)$$

$$i_X P^{j+1} = S_X \cdot P^j, \quad 0 \leq j \leq s+1. \quad (22)$$

Moreover, by the same arguments used in [10] one sees that S is a linear map.

Lemma 6.6 *Let $B \in H(r, s)$, then $\text{Ad}_B(S_X) = S_{BX}$ for every $X \in V$.*

Proof. By the definition of $H(r, s)$ and (21) and (22) we have for $0 \leq i \leq r$ and $0 \leq j \leq s$

$$\begin{aligned} R_{XZ_1 \dots Z_{i+4}}^{i+1} &= (B \cdot R^{i+1})_{XZ_1 \dots Z_{i+4}} \\ &= R_{B^{-1}XB^{-1}Z_1 \dots B^{-1}Z_{i+4}}^{i+1} \\ &= \left(S_{B^{-1}X} \cdot R^i \right)_{B^{-1}Z_1 \dots B^{-1}Z_{i+4}} \\ &= - \sum_{\alpha} R_{B^{-1}Z_1 \dots S_{B^{-1}X}B^{-1}Z_{\alpha} \dots B^{-1}Z_{i+4}}^i \\ &= - \sum_{\alpha} R_{B^{-1}Z_1 \dots B^{-1}BS_{B^{-1}X}B^{-1}Z_{\alpha} \dots B^{-1}Z_{i+4}}^i \\ &= - \sum_{\alpha} (B \cdot R^i)_{Z_1 \dots \text{Ad}_B(S_{B^{-1}X})Z_{\alpha} \dots Z_{i+4}} \\ &= - \sum_{\alpha} R_{Z_1 \dots \text{Ad}_B(S_{B^{-1}X})Z_{\alpha} \dots Z_{i+4}}^i \\ &= \left(\text{Ad}_B(S_{B^{-1}X}) \cdot R^i \right)_{Z_1 \dots Z_{i+4}}. \end{aligned}$$

On the other hand $i_X R^{i+1} = S_X \cdot R^i$, so that $\text{Ad}_B(S_{B^{-1}X}) \cdot R^i - S_X$ belongs to \mathfrak{h} . Since S_X belongs to \mathfrak{n} which is $\text{Ad}(H(r, s))$ -invariant, we also have that $\text{Ad}_B(S_{B^{-1}X}) \cdot R^i - S_X$ belongs to \mathfrak{n} . This implies that $\text{Ad}_B(S_{B^{-1}X}) \cdot R^i - S_X = 0$. ■

Corollary 6.7 *Let $A \in \mathfrak{h}$, then $A \cdot S = 0$.*

We take

$$\begin{aligned} T_X Y &= S_Y X - S_X Y, \\ K_{XY} &= R_{XY}^0 + [S_X, S_Y] + S_{T_X Y}, \\ P &= P^0. \end{aligned}$$

Proposition 6.8 *The triple (T, K, P) is an infinitesimal model.*

Proof. We have to show that (T, K, P) satisfies (13),..., (20). For (13), (14), (15), (19) and (20) one uses exactly the same arguments used in [10]. For the remaining, we observe that

$$\begin{aligned} R_{XY}^{i+2} - R_{YX}^{i+2} &= ([S_X, S_Y] + S_{T_X Y}) \cdot R^i, & 0 \leq i \leq r, \\ P_{XY}^{j+2} - P_{YX}^{j+2} &= ([S_X, S_Y] + S_{T_X Y}) \cdot P^j, & 0 \leq j \leq s. \end{aligned}$$

In fact, by (21)

$$\begin{aligned} R_{XY Z_1 \dots Z_{i+4}}^{i+2} &= (i_X R^{i+2})_{Y Z_1 \dots Z_i} = (S_X \cdot R^{i+1})_{XY Z_1 \dots Z_{i+4}} \\ &= -R_{S_X Y Z_1 \dots Z_{i+4}}^{i+1} - \sum_{\alpha=1}^{i+4} R_{Y Z_1 \dots S_X Z_{\alpha} \dots Z_{i+4}}^{i+1} \\ &= -\left(i_{S_X Y} R^{i+1}\right)_{Z_1 \dots Z_{i+4}} - \sum_{\alpha=1}^{i+4} \left(i_Y R^{i+1}\right)_{Z_1 \dots S_X Z_{\alpha} \dots Z_{i+4}} \\ &= -\left(S_{S_X Y} \cdot R^i\right)_{Z_1 \dots Z_{i+4}} - \sum_{\alpha=1}^{i+4} \left(S_Y \cdot R^i\right)_{Z_1 \dots S_X Z_{\alpha} \dots Z_{i+4}} \\ &= \sum_{\alpha=1}^{i+4} R_{Z_1 \dots S_{S_X Y} Z_{\alpha} \dots Z_{i+4}}^i + \sum_{\alpha, \beta=1}^{i+4} R_{Z_1 \dots S_X Z_{\alpha} \dots S_Y Z_{\beta} \dots Z_{i+4}}^i, \end{aligned}$$

and by (22) a similar argument holds for P_{XY}^{j+2} . Skew-symmetrizing in X, Y we obtain the desired formulae. Therefore, by (8) and (9) and the definition of K we obtain that $K \cdot R^i = 0$ and $K_{XY} \cdot P^j = 0$, for $0 \leq i \leq r$ and $0 \leq j \leq s$, so in particular $K_{XY} \cdot P^0 = 0$ and $K_{XY} \cdot R^0$. Making use of (11) this implies that $K_{XY} \in \mathfrak{h}$, whence $K_{XY} \cdot S = 0$ by Corollary 6.7, giving that $K_{XY} \cdot T = 0$. Finally, as a straightforward computation shows, for $A \in \mathfrak{h}$

$$(A \cdot K)_{XY} = (A \cdot R^0)_{XY} + [(A \cdot S)_X, S_Y] - [(A \cdot S)_Y, S_X] + S_{(A \cdot T)_{XY}}, \quad (23)$$

so that $K_{XY} \cdot K = 0$. ■

Proposition 6.9

$$\mathfrak{h} = \mathfrak{h}_0 = \{A \in \mathfrak{so}(V) / A \cdot K = 0, A \cdot T = 0, A \cdot P = 0\}.$$

Proof. Let $A \in \mathfrak{h}$, by Corollary 6.7 we have $A \cdot S = 0$, which implies $A \cdot T = 0$. In addition, by (23) we have $A \cdot K = 0$. Since $P = P^0$, by definition we deduce that $A \in \mathfrak{h}_0$, hence $\mathfrak{h} \subset \mathfrak{h}_0$. Conversely, let $A \in \mathfrak{h}_0$. We have that $A \cdot S = 0$ since S is recovered from T making use of

$$2\langle S_X Y, Z \rangle = -\langle T_X Y, Z \rangle + \langle T_Y Z, X \rangle - \langle T_Z X, Y \rangle.$$

On the other hand, by (23) we obtain $A \cdot R^0 = 0$, and since $P = P^0$ we also have $A \cdot P^0 = 0$. Now, a simple computation (see Lemma 6.6) shows that

$$\begin{aligned} (A \cdot R^{i+1})_X &= [A, S_X] \cdot R^i - S_{AX} \cdot R^i + S_X \cdot (A \cdot R^i) \\ &= (A \cdot S)_X \cdot R^i + S_X \cdot (A \cdot R^i), \quad 0 \leq i \leq r+1, \\ (A \cdot P^{j+1})_X &= [A, S_X] \cdot P^j - S_{AX} \cdot P^j + S_X \cdot (A \cdot P^j) \\ &= (A \cdot S)_X \cdot P^j + S_X \cdot (A \cdot P^j), \quad 0 \leq j \leq s+1. \end{aligned}$$

Using these formulae, by an inductive argument on the indices i and j we obtain that $A \cdot R^i = 0$ and $A \cdot P^j = 0$ for $0 \leq i \leq r+1$ and $0 \leq j \leq s+1$. Hence $A \in \mathfrak{h}$, proving that $\mathfrak{h}_0 \subset \mathfrak{h}$. ■

Definition 6.10 *The infinitesimal data $R^0, \dots, R^{r+2}, P^0, \dots, P^{s+2}$ is said regular if the associated infinitesimal model (T, K, P) is regular.*

Remark 6.11 *$R^0, \dots, R^{r+2}, P^0, \dots, P^{s+2}$ is recovered from the infinitesimal model (T, K, P) in the following way. As we have seen S is obtained from T by*

$$2\langle S_X Y, Z \rangle = -\langle T_X Y, Z \rangle + \langle T_Y Z, X \rangle - \langle T_Z X, Y \rangle.$$

With T and S one recovers R^0 using the definition of K . Finally, knowing R^0 and $P^0 = P$, and using (21) and (22), one can subsequently obtain R^i and P^j .

We are now in position to prove Theorem 6.2.

Proof of Theorem 6.2. Suppose that the infinitesimal model (T, K, P) associated with the infinitesimal data $R^0, \dots, R^{r+2}, P^0, \dots, P^{s+2}$ is regular. We consider the Nomizu construction $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus V$, and the Lie groups G_0 and H_0 , where G_0 is the simply-connected Lie group with Lie algebra \mathfrak{g}_0 and H_0 is its connected Lie subgroup with Lie algebra \mathfrak{h}_0 . Since H_0 is closed in G_0 we consider the homogeneous space G_0/H_0 , which is a reductive homogeneous space with reductive decomposition $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus V$. Identifying $V = T_o G_0/H_0$ we extend \langle, \rangle and P to a G_0 -invariant Riemannian metric g and a G_0 -invariant tensor field \bar{P} on G_0/H_0 respectively. We consider the canonical connection associated with that reductive decomposition (see [6, Ch. X]), which is an ASK-connection whose curvature and torsion coincide with K and T . As a straightforward computation using the properties of the canonical connection shows, $R^0, \dots, R^{r+2}, P^0, \dots, P^{s+2}$ coincide with the covariant derivatives of the curvature of g and \bar{P} at the origin $o \in G_0/H_0$. By the identification of $T_o G_0/H_0$ with V , we have that G_0/H_0 is (r, s) -strongly reductive. This proves the second part of the theorem.

Concerning the first part, we adapt the arguments used in [14]. Let (T, K, P) be the infinitesimal model associated with the infinitesimal data $R^0, \dots, R^{r+2}, P^0, \dots, P^{s+2}$, which now need not be regular. We consider the corresponding Nomizu construction $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus V$. Let G_0 be the simply-connected Lie group with Lie algebra \mathfrak{g}_0 , we choose an orthonormal

basis $\{e_1, \dots, e_n\}$ of V , and denote by $\{e^1, \dots, e^n\}$ its dual basis. Let $\{A_1, \dots, A_d\}$ be a basis of \mathfrak{h}_0 , and $\{A^1, \dots, A^d\}$ its dual basis. We write

$$\begin{aligned} T &= T_{\alpha\beta}^{\gamma} e^{\alpha} \otimes e^{\beta} \otimes e_{\gamma}, \\ K &= K_{\alpha\beta\gamma}^{\delta} e^{\alpha} \otimes e^{\beta} \otimes e^{\gamma} \otimes e_{\delta}, \\ P &= P_{\alpha_1 \dots \alpha_u}^{\beta_1 \dots \beta_v} e^{\alpha_1} \otimes \dots \otimes e^{\alpha_u} \otimes e_{\beta_1} \otimes \dots \otimes e_{\beta_v}, \end{aligned}$$

and define

$$\omega_{\beta}^{\alpha} = e^{\alpha}(A_{\gamma}(e_{\beta})) \otimes A^{\gamma},$$

where Einstein's summation convention is used. Note that $\omega_{\beta}^{\alpha} \in \mathfrak{g}^*$, so that

$$\omega = \omega_{\beta}^{\alpha} A_{\alpha} \otimes A^{\beta}$$

defines a left invariant 2-form on G_0 with values in $\mathfrak{h}_0 \subset \mathfrak{so}(V)$. Making use of the brackets defined in \mathfrak{g}_0 we easily obtain

$$de^{\alpha} = \frac{1}{2} T_{\beta\gamma}^{\alpha} e^{\beta} \wedge e^{\gamma}, \quad (24)$$

$$d\omega_{\beta}^{\alpha} = -\frac{1}{2} K_{\gamma\delta\beta}^{\alpha} e^{\gamma} \wedge e^{\delta} - \omega_{\gamma}^{\alpha} \wedge \omega_{\beta}^{\gamma}. \quad (25)$$

We now consider a coordinate system $\phi = (x^1, \dots, x^n, y^1, \dots, y^d)$ around the identity element $e \in G_0$ such that $dx|_e^{\alpha} = e|_e^{\alpha}$, and take

$$\begin{aligned} f : \quad \tilde{\mathcal{U}} &\rightarrow \mathcal{U} \\ (a_1, \dots, a_n) &\mapsto \phi^{-1}(a_1, \dots, a_n, 0, \dots, 0), \end{aligned}$$

where \mathcal{U} is the coordinate neighborhood and $\tilde{\mathcal{U}}$ is an open subset of \mathbb{R}^n where f can be defined. It is evident that the map f defines an immersion from an open set $W \subset \mathbb{R}^n$ containing the origin of \mathbb{R}^n into G_0 . Let $\tilde{E}^{\alpha} = f^*(e^{\alpha})$, since these 1-forms are linearly independent at the origin of \mathbb{R}^n , there is an open set $M \subset W$ around the origin where they are linearly independent. Let $\{\tilde{E}_1, \dots, \tilde{E}_n\}$ be the dual frame field, we define on M the pseudo-Riemannian metric

$$g = \sum_{\alpha=1}^n \tilde{E}^{\alpha} \otimes \tilde{E}^{\alpha},$$

and the tensor fields

$$\begin{aligned} \tilde{T} &= T_{\alpha\beta}^{\gamma} \tilde{E}^{\alpha} \otimes \tilde{E}^{\beta} \otimes \tilde{E}_{\gamma}, \\ \tilde{K} &= K_{\alpha\beta\gamma}^{\delta} \tilde{E}^{\alpha} \otimes \tilde{E}^{\beta} \otimes \tilde{E}^{\gamma} \otimes \tilde{E}_{\delta}, \\ \tilde{P} &= P_{\alpha_1 \dots \alpha_u}^{\beta_1 \dots \beta_v} \tilde{E}^{\alpha_1} \otimes \dots \otimes \tilde{E}^{\alpha_u} \otimes \tilde{E}_{\beta_1} \otimes \dots \otimes \tilde{E}_{\beta_v}. \end{aligned}$$

In addition we consider $\tilde{\omega} = f^*\omega$ which is a 1-form on M with values in \mathfrak{h}_0 . Note that $\{\tilde{E}_1, \dots, \tilde{E}_n\}$ is an orthonormal frame field defined on the whole M , so that it is a section of the bundle of orthonormal frames $\mathcal{O}(M)$ which trivializes it. Hence, making use of that section, $\tilde{\omega}$ is the 1-form of a metric connection $\tilde{\nabla}$ on $\mathcal{O}(M)$. By (24) and (25), which are

nothing but the structure equations for the torsion and curvature of ω , we have that \tilde{T} and \tilde{K} are the torsion and curvature of the connection $\tilde{\nabla}$ respectively. Since $\tilde{\omega}$ takes values in \mathfrak{h}_0 , we have that \tilde{T} , \tilde{K} and \tilde{P} are parallel with respect to $\tilde{\nabla}$, that is, $\tilde{\nabla}$ is an ASK-connection. Therefore, (M, g, \tilde{P}) is locally homogeneous with \tilde{P} invariant. Finally, making use of Remark 6.11, it is easy to see that the covariant derivatives of \tilde{P} and the curvature of g at the origin coincide with $R^0, \dots, R^{r+2}, P^0, \dots, P^{s+2}$ under the identification $T_o M \simeq V$. In addition, by this identification M is (r, s) -strongly reductive.

In both, the first and the second part of the theorem, uniqueness (up to local isometry) follows from Proposition 6.1. ■

Note that the strong reductivity condition (12) is essential in the proof of Theorem 6.2, since otherwise we are not able to construct the infinitesimal model (T, K, P) from the infinitesimal data $R^0, \dots, R^{r+2}, P^0, \dots, P^{s+2}$. This means that in general a locally homogeneous pseudo-Riemannian manifold whose metric is not definite might not be recovered from infinitesimal data. If the manifold admits an ASK-connection $\tilde{\nabla}$, this problem can be solved if we add to $R^0, \dots, R^{r+2}, P^0, \dots, P^{s+2}$ the knowledge of either S_p , where $S = \tilde{\nabla} - \nabla$, the torsion of $\tilde{\nabla}$ at p , or the curvature of $\tilde{\nabla}$ at p (these three last items provide equivalent information in view of Remark 6.11). In that case, an analogous result to Proposition 6.1 can be proved by a straightforward adaptation.

7 Examples and the reductivity condition

We begin this section showing a necessary condition for a reductive locally homogeneous pseudo-Riemannian manifold to be locally isometric to a globally homogeneous pseudo-Riemannian manifold. This question has already been solved in the Riemannian case (see for instance [9] and [12]).

Proposition 7.1 *Let (M, g, \mathcal{G}) be a reductive locally homogeneous pseudo-Riemannian manifold endowed with an associated AS-connection $\tilde{\nabla}$. If the infinitesimal model associated with $\tilde{\nabla}$ is regular, then (M, g) is locally isometric to a reductive globally homogeneous pseudo-Riemannian manifold. The same holds if the transvection algebra is regular.*

Proof. Let $p \in M$, consider the Nomizu construction $\mathfrak{g}_0 = T_p M \oplus \mathfrak{h}_0$ associated with the infinitesimal model (T, K) . Let G_0 be the simply-connected Lie group with Lie algebra \mathfrak{g}_0 , and H_0 its connected subgroup with Lie algebra \mathfrak{h}_0 . If (T, K) is regular then H_0 is closed in G_0 , so that we can consider the homogeneous space G_0/H_0 . Moreover, G_0/H_0 is reductive as $\mathfrak{g}_0 = T_p M \oplus \mathfrak{h}_0$ is a reductive decomposition, and the tangent space of G_0/H_0 at the origin o is identified with $T_p M$ through a linear isomorphism $F : T_p M \rightarrow T_o(G_0/H_0)$. This homogeneous space is thus endowed with a G_0 -invariant pseudo-Riemannian metric inherited from g at p . We consider the canonical connection $\tilde{\nabla}^{can}$ associated with this reductive decomposition (see [6, Ch. X]). Under the identification F , the

curvature and torsion of $\tilde{\nabla}$ coincides with K and T respectively. This means that there is a linear isometry $F : T_p M \rightarrow T_0(G_0/H_0)$ preserving the curvature and torsion of $\tilde{\nabla}$ and $\tilde{\nabla}^{can}$. Therefore, there are open neighborhoods \mathcal{U} and \mathcal{V} of p and o respectively, and an affine map $f : \mathcal{U} \rightarrow \mathcal{V}$ with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^{can}$ taking p to o (see [6, Vol. I, Ch. VI]). Since both connections are metric we have that f is an isometry. The same arguments can be applied substituting the Nomizu construction by the transvection algebra. ■

As we know, a globally homogeneous space can be represented as different coset spaces G/H . In the same way, we can consider the action of different Lie pseudo-groups of isometries on the same locally homogeneous pseudo-Riemannian manifold (M, g) . Since the notion of reductivity is tied to the action of a Lie pseudo-group in particular, the following question naturally arises: let \mathcal{G} and \mathcal{G}' be Lie pseudo-groups of isometries acting transitively on (M, g) , is it possible that (M, g, \mathcal{G}) is reductive but (M, g, \mathcal{G}') is non-reductive? We now present some examples which give an affirmative answer to this question, and explores the possible scenarios when \mathcal{G} is a subgroup of \mathcal{G}' and viceversa. We will also show that the reductivity condition does not imply the strong reductivity condition. It is worth pointing out that this situation is not a consequence of the freedom obtained by enlarging the (rather rigid) family of globally homogeneous spaces to the family of locally homogeneous spaces, and we can find illustrative examples restricting ourselves to globally homogeneous pseudo-Riemannian manifolds. We will finally give an example of an stabilizing pair distinct of $(k(p), l(p))$ (see Remark 5.1).

Example 7.2 Consider \mathbb{R}^5 endowed with the standard metric η of signature $(2, 3)$. We take the 4-dimensional submanifold

$$\mathbb{H}_1^4 = \{x \in \mathbb{R}^5 / \eta(x, x) = -1\},$$

endowed with the pseudo-Riemannian metric g inherited from η . (\mathbb{H}_1^4, g) is a Lorentz space of constant sectional curvature, and it is well known that it is the (globally) symmetric space

$$\mathbb{H}_1^4 \simeq \frac{SO_0(2, 3)}{SO_0(1, 3)}.$$

Let $\{e_1, \dots, e_5\}$ be the standard basis of \mathbb{R}^5 , and let e_i^j denote the endomorphism $e^j \otimes e_i$ of \mathbb{R}^5 . The isotropy algebra at the point $p = (0, 1, 0, 0, 0) \in \mathbb{H}_1^4$ is

$$\mathfrak{so}(1, 3) = \text{Span}\{e_1^3 + e_3^1, e_1^4 + e_4^1, e_1^5 + e_5^1, e_3^4 - e_4^3, e_3^5 - e_5^3, e_4^5 - e_5^4\}.$$

An $SO_0(1, 3)$ -invariant complement is

$$\mathfrak{m} = \text{Span}\{e_1^2 - e_2^1, e_2^3 + e_3^2, e_2^4 + e_4^2, e_2^5 + e_5^2\},$$

hence $(\mathbb{H}_1^4, g, SO_0(2, 3))$ is reductive. Consider now the Lie subalgebra \mathfrak{g} spanned by the elements

$$e_1^4 + e_4^1 - e_2^3 - e_3^2, \frac{1}{2}(e_1^2 - e_2^1 + e_1^3 + e_3^1 + e_2^4 + e_4^2 + e_4^3 - e_3^4),$$

$$\frac{1}{2}(e_1^3 + e_3^1 + e_2^1 - e_1^2 + e_2^4 + e_4^2 + e_3^4 - e_4^3), \frac{1}{2}(e_2^1 - e_1^2 + e_2^4 + e_4^2 + e_4^3 - e_3^4 - e_1^3 - e_3^1),$$

$$\frac{1}{\sqrt{2}}(e_1^5 + e_5^1 + e_5^4 - e_4^5), \frac{1}{\sqrt{2}}(e_3^5 - e_5^3 - e_2^5 - e_5^2), e_1^4 + e_4^1 + e_2^3 + e_3^2.$$

The isotropy algebra \mathfrak{k} at p is spanned by the elements

$$2(e_1^4 + e_4^1 + e_1^4), e_1^3 + e_3^1 + e_3^4 - e_4^3, \frac{1}{\sqrt{2}}(e_1^5 + e_5^1 + e_5^4 - e_4^5).$$

Let G be the connected Lie subgroup of $SO_0(2, 3)$ with Lie algebra \mathfrak{g} , then G acts transitively on \mathbb{H}_1^4 , but there is no $\text{ad}(\mathfrak{k})$ -invariant complement of \mathfrak{k} , so that (\mathbb{H}_1^4, g, G) is non-reductive (see Lie algebra $A5^*$ in [3]).

Example 7.3 We consider \mathbb{R}^4 endowed with the pseudo-Riemannian metric

$$g = 2e^{y_1} \cos y_2 (dy_1 dy_4 - dy_2 dy_3) - 2e^{y_1} \sin y_2 (dy^1 dy^3 + dy^2 dy^4) + Le^{4y_1} dy_2 dy_2,$$

with $L \in \mathbb{R} - \{0\}$. Let $\widetilde{SL(2, \mathbb{R})}$ be the universal cover of $SL(2, \mathbb{R})$, the group $G' = \widetilde{SL(2, \mathbb{R})} \ltimes \mathbb{R}^2 \times \mathbb{R}$ acts transitively by isometries on (\mathbb{R}^4, g) (see §5 of [3]). The Lie algebra of G' can be written as

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = e_4,$$

$$[e_1, e_5] = -e_5, \quad [e_2, e_5] = e_4, \quad [e_3, e_4] = e_5,$$

with respect to some basis $\{e_1, \dots, e_6\}$. It can be found as the Lie algebra $B3$ in [3], and moreover it is the full isometry algebra of (\mathbb{R}^4, g) and can be realized by the complete Killing vector fields

$$Y_1 = \cos(2y_2) \partial_{y_1} - \sin(2y_2) \partial_{y_2} + y_3 \partial_{y_3} - y_4 \partial_{y_4},$$

$$Y_2 = \frac{1}{2} \sin(2y_2) \partial_{y_1} + \cos^2(y_2) \partial_{y_2} + y_3 \partial_{y_4},$$

$$Y_3 = \frac{1}{2} \sin(2y_2) \partial_{y_1} - \sin^2(y_2) \partial_{y_2} + y_4 \partial_{y_3},$$

$$Y_4 = \partial_{y_4},$$

$$Y_5 = -\partial_{y_3},$$

$$Y_6 = e^{y_1} \cos(y_2) \partial_{y_3} + e^{y_1} \sin(y_2) \partial_{y_4}.$$

The isotropy algebra at $(0, 0, 0, 0) \in \mathbb{R}^4$ is $\text{Span}\{e_3, e_5 + e_6\}$. As stated in [3], (\mathbb{R}^4, g, G') is non-reductive. Let $\mathfrak{g} = \text{Span}\{e_1, e_2, e_4, e_5, e_6\}$. Making use of the distribution generated by the corresponding Killing vector fields we see that the action of the connected Lie subgroup $G \subset G'$ with Lie algebra \mathfrak{g} is still transitive. The isotropy algebra at $(0, 0, 0, 0)$ is $\mathfrak{k} = \text{Span}\{e_5 + e_6\}$, and $\mathfrak{m} = \text{Span}\{e_1, e_2, e_4, e_5\}$ is an $\text{Ad}(K)$ -invariant complement, where $K \subset G$ is the isotropy group with respect to the action of G at $(0, 0, 0, 0)$. Therefore (\mathbb{R}^4, g, G) is reductive. On the other hand we can check that (\mathbb{R}^4, g) is not strongly reductive. In this case, since there is no extra geometric structure, the complex of filtrations reduces to

$$\mathfrak{so}(T_p M) \supset \mathfrak{g}(p, 0) \supset \mathfrak{g}(p, 1) \supset \dots$$

A simple computation shows that the only non-zero component of the curvature is $R_{\partial_{y_1}\partial_{y_2}\partial_{y_1}\partial_{y_2}} = -3Le^{4y_1}$, and $\nabla R = 0$. We take $p = (0, 0, 0, 0)$ and $L = 1$ for the sake of simplicity, so that the filtration actually is

$$\mathfrak{so}(T_p M) \supset \mathfrak{g}(p, 0) = \mathfrak{g}(p, 1),$$

where

$$\mathfrak{so}(T_p M) = \left\{ \begin{pmatrix} -e & 2(b-c) & b & 0 \\ f & 2a & a & c \\ 2(d-f) & 0 & -2a & 2(b-c) \\ 0 & 2(d-f) & d & e \end{pmatrix} / a, b, c, d, e, f \in \mathbb{R} \right\},$$

$$\mathfrak{g}(p, 0) = \{A \in \mathfrak{so}(T_p M) / e = 2a, f = d\}.$$

It is easy to check that $\mathfrak{g}(p, 0)$ does not admit any complement \mathfrak{n} invariant by the adjoint action of $\mathfrak{g}(p, 0)$, hence (\mathbb{R}^4, g) cannot be strongly reductive.

We now exhibit an example of a locally homogeneous pseudo-Kähler manifold with an stabilizing pair distinct form (k, l) , where as usual (k, l) are the first integers such that $\mathfrak{g}(p, k) = \mathfrak{g}(p, k+1)$ and $\mathfrak{p}(p, l) = \mathfrak{p}(p, l+1)$.

Example 7.4 Consider the space \mathbb{C}^2 with complex coordinates (w, z) . We take $M = \mathbb{C}^2 - \{|w| = 0\}$ with the standard complex structure J and the pseudo-Riemannian metric

$$g = dw^1 dz^1 + dw^2 dz^2 + b(dw^1 dw^1 + dw^2 dw^2),$$

where $w = w^1 + iw^2$, $z = z^1 + iz^2$, and b is a function depending on w^1 and w^2 and satisfying $\Delta b = \frac{b_0}{||w||^4}$. This manifold is locally homogeneous since it admits an ASK-connection [2]. Let $\theta = -\frac{1}{||w||^2}(w^1 dw^1 + w^2 dw^2)$, the curvature tensor and its first covariant derivative are

$$R = \frac{1}{2} \frac{b_0}{||w||^4} (dw^1 \wedge dw^2 \otimes dw^1 \wedge dw^2), \quad \nabla R = 4\theta \otimes R.$$

We set $b_0 = 2$ and take the point $p = (-1, 0, 0, 0)$, so that

$$R_p = dw^1 \wedge dw^2 \otimes dw^1 \wedge dw^2, \quad \nabla R_p = 4dw^1 \otimes R_p,$$

$$\nabla^2 R_p = (20dw^1 \otimes dw^1 - 4dw^2 \otimes dw^2) \otimes R_p.$$

On the other hand, J_p is the standard complex structure of \mathbb{C}^2 and $\nabla J_p = 0$ since the manifold is pseudo-Kähler. A straightforward computations thus shows that the complex of filtrations is

$$\begin{array}{ccccccc} \mathfrak{so}(\mathbb{R}^4)^6 & \supset & \mathfrak{g}(p, 0)^2 & \supset & \mathfrak{g}(p, 1)^1 & = & \mathfrak{g}(p, 2)^1 \\ \cup & & \parallel & & \parallel & & \parallel \\ \mathfrak{p}(p, 0)^4 & \supset & \mathfrak{h}(p, 0, 0)^2 & \supset & \mathfrak{h}(p, 1, 0)^1 & = & \mathfrak{h}(p, 2, 0)^1 \\ \parallel & & \parallel & & \parallel & & \parallel \\ \mathfrak{p}(p, 1)^4 & \supset & \mathfrak{h}(p, 0, 1)^2 & \supset & \mathfrak{h}(p, 1, 1)^1 & = & \mathfrak{h}(p, 2, 1)^1, \end{array}$$

where superindexes indicate dimension. We have that $(k, l) = (1, 0)$, but $(r, s) = (1, -1)$ is a stabilizing pair.

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